Infinite Paths in Planar Graphs V, 3-Indivisible Graphs

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Abstract

We prove Nash-Williams' conjecture that every 4-connected, 3-indivisible, infinite, planar graph contains a spanning 2-way infinite path. A graph is said to be 3-indivisible if the deletion of any finite set of vertices results in at most two infinite components.

1 Introduction

We use the terminology in [9, 10, 11, 12]. A graph G is *k-indivisible*, where k is a positive integer, if, for every finite $X \subseteq V(G)$, $G-X$ has at most $k-1$ infinite components. Nash-Williams [4, 5] (also see [7]) conjectured that a 4-connected infinite planar graph contains a spanning 2-way infinite path if, and only if, G is 3-indivisible. The goal of this paper is to prove the following result, completing the proof of the Nash-Williams conjecture started in [9].

(1.1) Theorem. *Every 4-connected, 3-indivisible, infinite, planar graph contains a spanning 2-way infinite path.*

Let C be a cycle in an infinite plane graph G. If C bounds a closed region such that the vertices and edges of G in that region form a finite subgraph of G, then we use $I_G(C)$ to denote this finite subgraph. C is said to be *dividing* in G if $I_G(C)$ is not defined. Let $\gamma(G)$ denote the maximum number of vertex disjoint dividing cycles in G.

In [9, 10, 12], Nash-Williams' conjecture is proved for 3-indivisible graphs G with $\gamma(G) = 0$ or $\gamma(G) = \infty$. In this paper, we prove the remaining case from which Theorem (1.1) follows.

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(1.2) Theorem. *Let G be a 4-connected, 3-indivisible, infinite, plane graph, and assume that $\gamma(G)$ is a positive integer. Then G contains a spanning 2-way infinite path.*

We mention that Bruhn (see [3]) conjectured the following generalization of Tutte's Hamilton cycle theorem: Every locally finite 4-connected planar graph admits a Hamilton circle. See [3] for the definitions of circle and Hamilton circle. This conjecture is verified for 6-connected graphs with finitely many ends [1], for graphs with exactly one end [9, 10], and for graphs with no dividing cycles [2].

This paper is organized as follows. First, we summarize those concepts and results in [9, 10, 11, 12] that will be used in our proof of Theorem (1.2). This is done in section 2. Additional lemmas concerning Tutte paths are proved in section 3. In section 4, we consider partial nets and show that certain partial nets can be used to extend Tutte subgraphs. The main proof of Theorem (1.2) starts with section 5, where we investigate the structure of 3-indivisible infinite plane graphs G for which $\gamma(G)$ is a positive integer. We choose an appropriate curve in the plane that separates the graph to two infinite graphs. This curve is then used to build partial nets which can be used to extend Tutte subgraphs. We show in section 6 that to prove Theorem (1.2) it suffices to prove the existence of a 2-way infinite Tutte path in a certain 3-indivisible plane graph. (This is done by using a tight partial net.) The solution of the Tutte path problem depends on the parity of the number of steps needed to reduce the new graph. In section 7 we take care of the odd case; and the even case is taken care of in section 8.

We conclude this section with some helpful terminology. Let H be a (finite or infinite) subgraph of a (finite or infinite) graph G, let $v_1, \ldots, v_k \in V(G)$, and $\{u_i, w_i\} \subseteq V(H) \cup \{v_1, \ldots, v_k\}$, $i = 1, \ldots, m$. Then $H + \{v_1, \ldots, v_k, u_1w_1, \ldots, u_mw_m\}$ denotes the graph with vertex set $V(H) \cup \{v_1, \ldots, v_k\}$ and edge set $E(H) \cup \{u_1w_1, \ldots, u_kw_k\}$. For any $x \in V(H) \cup E(H)$, we write $H + x$ instead of $H + \{x\}$.

A *separation* of a (finite or infinite) graph G is an ordered pair (G_1, G_2) of subgraphs of G such that $E(G_i) \neq E(G)$ for $i \in \{1, 2\}$, $E(G_1) \cap E(G_2) = \emptyset$, and $G_1 \cup G_2 = G$; and if $|V(G_1 \cap G_2)| = k$ then we say that (G_1, G_2) is a k-separation of G.

Let C be a cycle in a plane graph and let $x, y \in V(C)$ and $e \in E(C)$. If $x \neq y$ then xCy denotes the clockwise subpath of C from x to y, and if $x = y$ then xCy denotes the trivial path consisting of the vertex $x = y$. If x is incident with e then xCe is the trivial path consisting of x; if x is not incident e, then we use xCe to denote the subpath of $C - e$ from x to e in clockwise order on C. Also, for a (finite or infinite) path P and $x, y \in V(P)$, xPy denote the unique subpath of P between x and y.

2 Summary of previous results

First, we recall that a *net* in an infinite plane graph G is a sequence $N = (C_1, C_2, \ldots)$ of cycles in G such that (1) $I(C_i)$ is defined for all $i \geq 1$, (2) $I(C_i) \subseteq I(C_{i+1})$ for all

$i \geq 1$, (3) $\bigcup_{i=1}^{\infty} I(C_i) = G$, and (4) either $C_i \cap C_j = \emptyset$ for all $i \neq j$, or, for $i \geq 1$, $C_i \cap C_{i+1}$ is a non-trivial path, $C_i \cap C_{i+1} \subseteq C_{i+1} \cap C_{i+2}$, and neither endvertex of $C_i \cap C_{i+1}$ is an endvertex of $C_{i+1} \cap C_{i+2}$. If $C_i \cap C_j = \emptyset$ for all $i \neq j$, then N is called a *radial net*; otherwise, N is said to be a *ladder net*. Let $\partial N = \emptyset$ if N is a radial net; otherwise, let $\partial N = \bigcup_{i=1}^{\infty}(C_i \cap C_{i+1})$, which is a 2-way infinite path.

Let G be a plane graph, infinite or finite. If G is finite, let ∂G denote the maximal subgraph of G contained in the boundary of the infinite face of G. If G is infinite, let ∂G denote the subgraph of G such that for any $x \in V(G)$, $x \in V(\partial G)$ iff $x \notin V(I(C)) - V(C)$ for all nondividing cycles C in G, and for any $x \in E(G)$, $x \in E(\partial G)$ iff $x \notin E(I(C)) - E(C)$ for all nondividing cycles C in G. Note that if G is a 2-connected 2-indivisible infinite plane graph, then ∂G is a path, or a 1-way infinite path, or a 2-way infinite path.

For a (finite or infinite) subgraph H of a graph G, we say that G is $(4, H)$-connected if for any subset $S \subseteq V(G)$ with $|S| \leq 3$, every component of $G - S$ contains a vertex of H. The following is Lemma (2.2) in [12].

(2.1) Lemma. *Let G be a 2-connected infinite plane graph and C be a facial cycle of G such that G is $(4, C)$-connected. If G is 2-indivisible, then all but one face of G are bounded by cycles, and ∂G is precisely the subgraph of G lies on the boundary of the exceptional face of G.*

Let G be a (finite or infinite) graph and H be a (finite or infinite) subgraph of G. An *H-bridge* of G is a (finite or infinite) subgraph of G which is induced by either (1) an edge of $E(G) - E(H)$ whose incident vertices are in $V(H)$ or (2) the edges contained in a component of $G - V(H)$ and the edges from that component to H.

The following result is Theorem (2.1) in [11] (its 4-connected version is given in [10]), which describes the structure of 2-indivisible graphs.

(2.2) Theorem. *Let G be a 2-connected 2-indivisible infinite plane graph with a facial cycle C such that G is $(4, C)$-connected, and let S denote the set of vertices of G of infinite degree. Then $|S| \leq 2$, and there is a set F of edges of G such that*

(1) for any $f \in F$, f is incident with a vertex in S,

(2) $G - F$ has a net $N = (C_1, C_2, \ldots)$, $C \subseteq I(C_1)$, $S \subseteq \partial N$, and for any $f \in F$ both incident vertices of f are contained in a common infinite S-bridge of ∂N,

(3) if $|S| = 1$, then either one S-bridge of ∂N contains all vertices incident with edges in F or each S-bridge of ∂N contains infinitely many vertices incident with edges in F, and

(4) *if $|S| = 2$, then for any $T \subseteq V(G) - S$ with $|T| \leq 3$, S is contained in a component of $(G - F) - T$.*

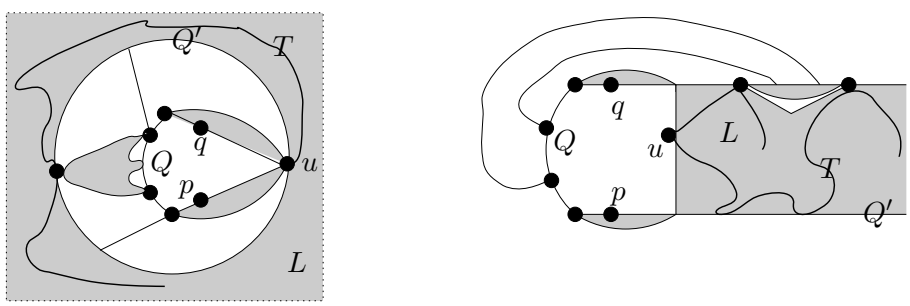
Let G be a (finite or infinite) graph and H be a (finite or infinite) subgraph of G. If B is an H-bridge of G, then the vertices in $V(H \cap B)$ are called *attachments* of B (on H). The subgraph H is a *Tutte subgraph* of G if every H-bridge of G is finite and has at most three attachments. For a (finite or infinite) subgraph C of G, we say that H is a *C-Tutte subgraph* of G if H is a Tutte subgraph of G and every H-bridge of G containing an edge of C has at most two attachments. A (finite or infinite) *Tutte path* is a (finite or infinite) path which is a Tutte subgraph.

The following two results are due to Thomassen [8] and Thomas and Yu [6], respectively.

(2.3) Lemma. *Let G be a finite 2-connected plane graph with a facial cycle C. Assume that $x \in V(C)$, $e \in E(C)$, and $y \in V(G - x)$. Then G contains a C-Tutte path P from x to y such that $e \in E(P)$.*

(2.4) Lemma. *Let G be a finite 2-connected plane graph with a facial cycle C. Let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in clockwise order. Then G contains a vCu-Tutte path P from u to v such that $\{e, f\} \subseteq E(P)$.*

Remark. Lemmas (2.3) and (2.4) may be applied when e or f or both are vertices. Also, when we apply these two lemmas, we shall allow G to be the complete graph on two vertices; in which case, we have $P = G$.



(a) Q' is a cycle

(b) Q' is a path or 2-way infinite path

Figure 1: Illustration of Lemma (2.5).

We need Lemma (3.3) from [11], which will be used to extend Tutte paths. See Figure 1 for an illustration.

(2.5) Lemma. *Let K be a connected (finite or infinite) plane graph, C a facial walk of K, Q a path between p and q on C, $u \in V(C) - V(Q)$, L a subgraph of $K - V(Q)$, and Q' a cycle in L or a path in L or a 2-way infinite path in L. Suppose the following three conditions are satisfied:*

(1) for any $(L \cup Q)$-bridge B of K, $|V(B \cap L)| \leq 1$ and $V(B \cap L) \subseteq V(Q')$,

(2) $K - V(L)$ is finite and all vertices of $K - V(L)$ have finite degrees in K, and

(3) L contains a Q'-Tutte subgraph T with $u \in V(T)$ and $|V(Q') \cap V(T)| \geq 2$.

Then $K - V(T)$ contains a path S between p and q such that $S \cup T$ is a Q-Tutte subgraph of K, and every T-bridge of L containing no edge of Q' is also an $(S \cup T)$-bridge of K.

The following result is Corollary (3.7) in [10].

(2.6) Lemma. *Let G be a 2-connected infinite plane graph with a ladder net N, and let $x \in V(\partial N)$ and $uv \in E(\partial N)$ such that $u \in V(x\partial Nv)$. Then G contains a 1-way infinite ∂N-Tutte path P from x such that $uv \in E(P)$ and $u \in V(xPv)$.*

We also need Theorem (1.2) from [11].

(2.7) Theorem. *Let G be a 2-connected 2-indivisible infinite plane graph, let C be a facial cycle of G, let $x \in V(C)$ and $uv \in E(C)$ with $x \neq v$, and let Q denote the subpath of $C - v$ between u and x. Assume that G is $(4, C)$-connected and v is contained in the infinite component of $G - V(Q)$. Then G contains a 1-way infinite C-Tutte path P from x such that $uv \in E(P)$ and $u \in V(xPv)$.*

3 Additional lemmas on Tutte paths

In this section, we prove several lemmas needed later. We begin with a variation of Lemma (2.6).

(3.1) Lemma. *Let G be a 2-connected 2-indivisible infinite plane graph with a ladder net N. Suppose that $x, y, v \in V(\partial N)$ and $v \notin V(x\partial Ny) - \{x, y\}$. Then G contains a 1-way infinite ∂N-Tutte path P from $\{x, y\}$ such that $\{x, y, v\} \subseteq V(P)$.*

Proof. Let d denote the length of the shortest subpath of ∂N between v and $\{x, y\}$. If $d = 0$, then $v \in \{x, y\}$ and the assertion of the lemma follows from Lemma (2.6). So we may assume $d \geq 1$. Without loss of generality, we may assume $y \in V(x\partial Nv)$. Let N_x (respectively, N_y) denote the infinite $\{x, y\}$-bridge of ∂N containing x (respectively, y).

Let $Q = x\partial Ny$, and let Q' be a path in $G - V(Q)$ from $x' \in V(N_x)$ to $y' \in V(N_y)$ internally disjoint from N_x and N_y. We choose Q' so that the subgraph $H := I(x'\partial Ny' \cup$

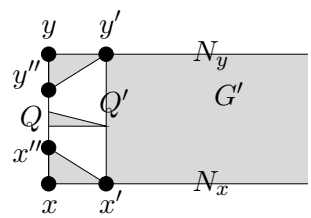


Figure 2: G, Q, G', Q', x and y

Q') is minimal. See Figure 2. Let $G' := G - (V(H) - V(Q'))$. Clearly, G' has a ladder net N' with $\partial N' = Q' \cup (\partial N \cap G')$. By the minimality of H, we see that every $(Q \cup G')$-bridge of G has at most one attachment on G', which must be on Q'.

Let $v' = v$ if $v \in V(G')$, and otherwise, let $v' = y'$. Note that the subpath of $\partial N'$ between v' and $\{x', y'\}$ either has length 0 or is a proper subpath of $v\partial Ny$. Hence by induction (on d), G' has a 1-way infinite $\partial N'$-Tutte path P' from $\{x', y'\}$ such that $\{x', y', v'\} \subseteq V(P')$. Moreover, if $v' = y'$, then by Lemma (2.6) we may choose P' so that P' is from y'. If $v' \neq y'$ then we may assume P' is from y' because in that case our argument below is symmetric with respect to x' and y'.

Let $x'', y'' \in V(Q)$ with $y''Qy$ and $x''Qx$ maximal such that $\{y', y''\}$ and $\{x', x''\}$ are each contained in a $(Q \cup G')$-bridge of G. Let X (respectively, Y) denote the union of $x''Qx$ (respectively, $y''Qy$) and those $(Q \cup G')$-bridges of G whose attachments are all contained in $x''Qx + x'$ (respectively, $y''Qy + y'$).

In $Y + y''y'$ we use Lemma (2.4) to find a $y'\partial Ny''$-Tutte path P_y from y' to y'' such that $y \in V(P_y)$ and $v \in V(P_y)$ (if $v' = y'$). In $X + x'x''$, we use Lemma (2.3) to find an $x'\partial Nx''$-Tutte path P'_x from x to x' such that $x'x'' \in E(P'_x)$; then $P_x := P'_x - x'$ is a path from x to x''.

Let $K := G - V((X \cup Y) - \{x', x'', y', y''\})$. With $K, G', x''Qy'', \partial N', P', x'$ as K, L, Q, Q', T, u, respectively, the conditions of Lemma (2.5) hold. So by Lemma (2.5), there is a path S from x'' to y'' such that $P' \cup S$ is an $x''Qy''$-Tutte subgraph of K and every P'-bridge of G' not containing an edge of $\partial N'$ is a $(P' \cup S)$-bridge of K. Because $x', y' \in V(P')$, we see that every P'-bridge of G' not containing an edge of Q' is a $(P' \cup S)$-bridge of K.

It is now easy to see that $P := P' \cup P_y \cup S \cup P_x$ is a ∂N-Tutte path of G from x and containing $\{y, v\}$. □

The next result is a variation of Lemma (2.4).

(3.2) Lemma. *Let G be a finite 2-connected plane graph, and let u, u', v', v be vertices on a facial cycle C of G in clockwise order. Suppose $u \neq u'$ and there is no 2-separation*

(G_1, G_2) *of* G *such that* $u'Cv' \subseteq G_1$ *and* $vCu \subseteq G_2$*. Then* G *contains a* $v'Cu'$*-Tutte (or* C*-Tutte if* $v' = u'$*) path* P *from* u *to* u' *and containing* $\{v, v'\}$*.*

Proof. For convenience, we may assume that G is embedded so that $C = \partial G$. See Figure 3. We may assume $v' \neq u'$; for otherwise, the assertion of the lemma follows from Lemma (2.3). Let v'' be the neighbor of v' in $u'Cv'$. Then $v'' \neq u'$, for, otherwise, G has a 2-separation (G_1, G_2) such that $u'Cv' = G_1$ and $vCu \subseteq G_2$, a contradiction. Let H denote the union of those blocks of $G - V(v'Cu)$, each of which contains an edge of $u'Cv''$. Let $u'' \in V(H \cap uCu')$ with uCu'' minimal. Let R denote the path in H induced by cut edges of H and edges in $E(\partial H) - E(u'Cv'')$.

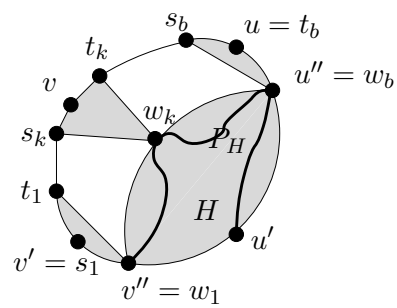


Figure 3: G and H

Let $w_1, \ldots, w_b$ denote the attachments on R of $(v'Cu \cup H)$-bridges of G, in order from v'' to u''. Then $w_1 = v''$ and $w_b = u''$. For each $1 \leq j \leq b$, let $s_j, t_j \in V(v'Cu)$ such that v', s_j, t_j, u occur on $v'Cu$ in order, $\{s_j, w_j\}$ and $\{t_j, w_j\}$ are each contained in a $(v'Cu \cup H)$-bridge of G, and subject to these conditions, s_jCt_i is maximal. For $j \in \{1, b\}$, let T_j denote the union of s_jCt_i and those $(v'Cu \cup H)$-bridges of G whose attachments are all contained in $s_jCt_j + w_j$.

Clearly, $v \in V(t_{k-1}Ct_k)$ for some $1 \leq k \leq b$, where $t_0 = v'$. Let T_k denote the union of $t_{k-1}Ct_k$ and those $(v'Cu \cup H)$-bridges of G whose attachments are all contained in $t_{k-1}Ct_k + w_k$. Note that $k \neq b$; for, otherwise, G has a 2-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{t_{b-1}, w_b\}$, $u'Cv' \subseteq G_1$, and $vCu \subseteq G_2$, a contradiction.

In $T_b + s_bu''$, we apply Lemma (2.4) to find an s_bCu''-Tutte path P'_b from u to u'' such that $s_bu'' \in E(P_b)$, and let $P_b := P'_b - u''$. If $k = 1$, we apply Lemma (2.4) to find a $v''Ct_1$-Tutte path P_1 in $T_1 + t_1v''$ from v'' to t_1 such that $\{v, v'\} \subseteq V(P_1)$. Now assume $k \neq 1$. By Lemma (2.3) we find a $v''Ct_1$-Tutte path P_1 in $T_1 + t_1v''$ from v'' to t_1 through v', and we apply Lemma (2.4) to find a $t_{k-1}Ct_k$-Tutte path P'_k in $T_k + t_{k-1}w_k$ from t_k to w_k such that $v \in V(P'_k)$ and $t_{k-1}w_k \in E(P'_k)$, and let $P_k := P'_k - w_k$.

By applying Lemma (2.4), we find an R-Tutte path P_H in $H + u'v''$ from u' to v'' such that $\{u'', w_k\} \subseteq V(P_H)$.

When $k = 1$, let $S' = \emptyset$. When $k \neq 1$, let K' be the union of H, t_1Ct_{k-1}, and those $(v'Cu \cup H)$-bridges of G whose attachments are all contained in $t_1Ct_{k-1} \cup H$. With $K', H, t_1Ct_{k-1}, R, P_H, w_k$ as K, L, Q, Q', T, u, respectively, the conditions of Lemma (2.5) hold. So by Lemma (2.5), there is a path S' in $K' - V(P_H)$ from t_1 to t_{k-1} such that $S' \cup P_H$ is a t_1Ct_{k-1}-Tutte subgraph of K', and every P_H-bridge of H not containing any edge of R is also an $(S' \cup P_H)$-bridge of K'.

Recall that $k \neq b$. Let K'' be the union of H, t_kCs_b, and those $(v'Cu \cup H)$-bridges of G whose attachments are all contained in $t_kCs_b \cup H$. With $K'', H, t_kCs_b, R, P_H, w_k$ as K, L, Q, Q', T, u, respectively, the conditions of Lemma (2.5) hold. Thus, by Lemma (2.5), there is a path S'' in $K''-V(P_H)$ from t_k to s_b such that $S''\cup P_H$ is a t_kCs_b-Tutte subgraph of K'', and every P_H-bridge of H not containing any edge of R is also an $(S''\cup P_H)$-bridge of K''.

It is easy to verify that $P := P_H \cup P_1 \cup S' \cup P_k \cup S'' \cup P_b$ is a $v'Cu'$-Tutte path in G from u to u' such that $\{v, v'\} \subseteq V(P)$. □

When we remove the 2-separation requirement from Lemma (3.2), we have a weaker conclusion (to be used in section 6).

(3.3) Lemma. *Let G be a finite 2-connected plane graph, and let u, u', v', v be vertices on a facial cycle C of G in clockwise order such that $u \neq u'$ or $u \neq v'$. Then there is a $v'Cu'$-Tutte (or C-Tutte if $v' = u'$) path P in G from u to $\{u', v'\}$ such that $\{u, v, u', v'\} \subseteq V(P)$.*

Proof. For convenience, we may assume that G is embedded in the plane so that $C = \partial G$. We may assume u, u', v', v are all distinct; for otherwise, the assertion of the lemma follows from Lemma (2.3). See Figure 3 for illustration. We may also assume that there is a 2-separation (G_1, G_2) of G such that $u'Cv' \subseteq G_1$ and $vCu \subseteq G_2$; otherwise, the assertion of the lemma follows from Lemma (3.2). Let $V(G_1 \cap G_2) = \{u'', v''\}$ such that $u'' \in V(uCu')$ and $v'' \in V(v'Cv)$. Moreover, we may select (G_1, G_2) such that G_2 is minimal.

Case 1. G_2 is 2-connected and $u \neq u''$.

Then by minimality of G_2, we may apply Lemma (3.2) to find a $v''Cu''$-Tutte path P_2 in G_2 from u to u'' such that $\{v'', v\} \subseteq V(P_2)$.

If $\{u'', v''\} = \{u', v'\}$, then $P := P_2$ gives the desired path. So we may assume that $\{u'', v''\} \neq \{u', v'\}$. We may further assume $\{u'', v''\} \cap \{u', v'\} = \emptyset$; for otherwise, we may apply Lemma (2.3) to find a $u''Cv''$-Tutte path P_1 in $G_1 + u''v''$ from v'' to $\{u', v'\} - \{u'', v''\}$ through $u''v''$, and it is easy to see that $P := (P_1 - v'') \cup P_2$ gives the desired path.

Let H denote the union of those blocks of $G_1 - v''$, each of which contains an edge of $u''Cv'$. Let $v^* \in V(H \cap v'Cv'')$ with v^*Cv'' minimal, and let $u^* \in V(H \cap u''Cu')$ with

u^*Cu' minimal. Let H^* denote the u^*-bridge of H containing u^*Cv^*, and let Q denote the path induced by cut edges of H^* and edges in $E(\partial H^*) - E(u^*Cu')$.

Suppose no vertex of $u'Cv' - v'$ is a cutvertex of H. Then $u^* \neq u'$ and H^* is a 2-connected plane graph. So we apply induction to find a $(v'Qu^* \cup u^*Cu')$-Tutte path P_1^* in H from u^* to $\{u', v'\}$ such that $\{u^*, u', v', v^*\} \subseteq V(P_1^*)$. Let $P^* = \{u^*\}$ if $u^* = u''$; otherwise, we apple Lemma (2.3) to the blocks of $H - V(H^* - u^*)$ and find a $u''Cu^*$-Tutte path P^* from u'' to u^*. Then $P_1 := P_1^* \cup P^*$ is a path from u'' to $\{u', v'\}$ containing $\{u'', u', v', v^*\}$. It is easy to see that $P := P_1 \cup P_2$ gives the desired path.

Now assume that some vertex of $u'Cv' - v'$ is a cut vertex of H. In this case, the block of H containing v' and v^* cannot contain any edge of $u''Cu'$. Hence, we may apply Lemma (2.3) (to each block of H) to find a Q-Tutte path P_1 in H from u'' to v' such that $v^*, u' \in V(P_1)$. Now $P := P_1 \cup P_2$ gives the desired path.

Case 2. G_2 is not 2-connected or $u = u''$.

By the minimality of G_2, all cut vertices of G_2 must be contained in $vCu - \{v, u\}$. Thus we may apply Lemma (2.3) to the blocks of G_2 and find a $v''Cu''$-Tutte path P_2 from u to v'' containing $\{v, u''\}$.

If $\{u'', v''\} = \{u', v'\}$, then $P := P_2$ gives the desired path. So we may assume that $\{u'', v''\} \neq \{u', v'\}$. We may further assume $\{u'', v''\} \cap \{u', v'\} = \emptyset$; for otherwise, we apply Lemma (2.3) to find a $u''Cv''$-Tutte path P_1 in $G_1 + u''v''$ from u'' to $\{u', v'\} - \{u'', v''\}$ through $u''v''$, and it is easy to see that $P := (P_1 - u'') \cup P_2$ gives the desired path.

Let H denote the union of those blocks of $G_1 - u''$, each of which contains an edge of $u'Cv''$. Let $u^* \in V(H \cap u''Cu')$ with $u''Cu^*$ minimal, and let $v^* \in V(H \cap v'Cv'')$ with $v'Cv^*$ minimal. Let H^* denote the v^*-bridge of H containing u^*Cv^*, and let Q denote the path induced by cut edges of H^* and edges in $E(\partial H^*) - E(v'Cv^*)$. By applying the same argument as in Case 1, with v'', v^*, v', u', u^* playing the roles of u'', u^*, u', v', v^*, respectively, we can show that there is a $(v'Cv^* \cup v^*Qu')$-Tutte path P_1^* in H^* from v^* to $\{v', u'\}$ containing $\{v^*, v', u', u^*\}$ and there is a v^*Cv''-Tutte path P^* in $H - V(H^* - v^*)$ from v^* to v''. Now $P_1 := P_1^* \cup P^*$ is a path in G_1 from v'' to $\{v', u'\}$ containing $\{v'', v', u', u^*\}$. It is easy to see that $P := P_1 \cup P_2$ gives the desired path. □

We conclude this section with a lemma which is needed to deal with graphs that are not locally finite. Its notation is chosen to facilitate its applications. See Figure 4 for illustration.

(3.4) Lemma. *Let B^* be a finite connected plane graph drawn in a closed disc in the plane, let b_1, b_2, x_1', x', x^* be vertices of B^* which are distinct except possibly $x' = x^*$, and let Q be a path in $B^* - \{b_1, b_2, x_1'\}$ between x' and x^*. Assume that $b_1, x_1', x', Q, x^*, b_2$ occur on the boundary of the disc in clockwise order, x_1' has a neighbor in $B^* - b_1$, and B^* is $(4, Q + \{x_1', b_1, b_2\})$-connected. Moreover, assume $B^* - V(Q + x_1')$ has a path from*

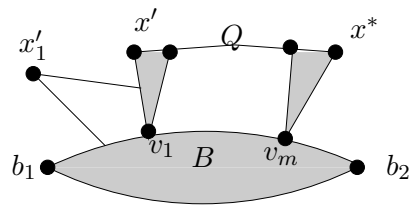


Figure 4: Illustration of Lemma (3.4).

b_1 to b_2, and $B^* - V((Q - x') + \{x'_1, b_2\})$ has a path from x' to b_1. Then one of the following holds:

(1) there is a path P_B in $B^* - \{x'_1, b_1, b_2\}$ from x' to x^* such that $P_B + \{x'_1, b_1, b_2\}$ is a Q-Tutte subgraph of B^*; or

(2) there is a path P_B in $B^* - \{b_1, b_2\}$ from x'_1 to x^* through x' such that $P_B + \{b_1, b_2\}$ is a Q-Tutte subgraph of B^*.

Proof. We may assume $x^* \neq x'$. For, suppose $x^* = x'$. By Lemma (2.4), we find a Tutte path P'_B from b_1 to b_2 in $B^* + \{b_1x'_1, b_2x^*\}$ containing $b_1x'_1$ and b_2x^*. Now $P_B := P'_B - \{b_1, b_2\}$ gives the desired path for (2).

Let B denote the minimal union of blocks of $(B^* - x'_1) - V(Q)$ containing a path from b_1 to b_2. Note that every $(B \cup Q)$-bridge of $B^* - x'_1$ has at most one attachment in B. Let $v_1, \ldots, v_m$ be the attachments on B of $(B \cup Q)$-bridges of $B^* - x'_1$, and assume that $b_1, v_1, \ldots, v_m, b_2$ occur on ∂B in order. See Figure 4.

We may assume $v_1 \neq b_1$. For, suppose $v_1 = b_1$. Let $x'' \in V(Q)$ with $x'Qx''$ maximal such that $\{b_1, x'\}$ is contained in a $(B \cup Q)$-bridge of $B^* - x'_1$. Let (B', B'') denote a separation of B^* such that $V(B' \cap B'') = \{b_1, x''\}$, $\{x'_1, x'\} \subseteq V(B')$ and $\{x^*, b_2\} \subseteq V(B'')$. We apply Lemma (2.4) to $B'' + \{b_1x'', x^*b_2\}$ to find an $x''Qx^*$-Tutte path P'' from b_1 to b_2 containing b_1x'' and x^*b_2, and apply Lemma (2.4) to $B' + b_1x''$ to find an $x'Qx''$-Tutte path P' from x'_1 to b_1 containing x' and b_1x''. It is now easy to see that $P_B := (P' - b_1) \cup (P'' - \{b_1, b_2\})$ gives the desired path for (2).

We may also assume $v_m \neq b_2$. For otherwise, let $x'' \in V(Q)$ with $x''Qx^*$ maximal such that $\{x'', b_2\}$ is contained in a $(B \cup Q)$-bridge of $B^* - x'_1$. Let B'' denote the union of $x''Qx^*$ and those $(B \cup Q)$-bridges of $B^* - x'_1$ whose attachments are all contained in $x''Qx^* + b_2$. In $B'' + b_2x''$ we apply Lemma (2.3) to find an $x''Qx^*$-Tutte path P'' from x^* to b_2 through b_2x''. Let B' denote the $\{x'', b_2\}$-bridge of B^* containing $\{x'_1, b_1\}$. Applying induction to B' (with x'' playing the role of x^*), we find a path P' such that P' is a path from x'' to x' in $B' - \{x'_1, b_1, b_2\}$ and $P' + \{x'_1, b_1, b_2\}$ is an $x'Qx''$-Tutte subgraph of B', or P' is a path from x'' to x'_1 through x' in $B' - \{b_1, b_2\}$ and $P' + \{b_1, b_2\}$

is an $x'Qx''$-Tutte subgraph of B'. Then $P_B := P' \cup (P'' - b_2)$ gives the desired path for (1) or (2).

We may further assume $v_m \neq v_1$. For, suppose $v_m = v_1$. Let B' and B'' denote the $\{x'_1, v_1\}$-bridges of B^* containing B and Q, respectively. Suppose x'_1 has no neighbor in $B - b_1$. Then, since B^* is $(4, Q + \{x'_1, b_1, b_2\})$-connected, no cut vertex of $B'' - x'_1$ separates Q from v_1. In $B'' - x'_1$ we apply Lemma (2.3) to find a $\partial(B'' - x'_1)$-Tutte path P_B from x' to x^* through v_1, which gives the desired path for (1). Now assume that x'_1 has a neighbor in $B - b_1$. By Lemma (2.4) we find a Tutte path P' in $B' + \{b_1x'_1, v_1b_2\}$ from b_1 to b_2 through $b_1x'_1$ and v_1b_2. By Lemma (2.3) we find a $\partial(B'' - x'_1)$-Tutte path P'' in $(B'' - x'_1) + x^*v_1$ from x^* to v_1 through x'. Then $P_B := (P' - \{b_1, b_2\}) \cup P''$ gives the desired path for (2).

Let $B_1, \ldots, B_k$ denote the blocks of $B - \{b_1, b_2\}$. By planarity and since B^* is $(4, Q+\{x'_1, b_1, b_2\})$-connected, we may choose appropriate notation so that $B_i \cap B_j = \emptyset$ for $1 \leq i \leq j-2 \leq k-2$, $V(B_i \cap B_{i+1}) = \{w_i\}$ for $1 \leq i \leq k-1$, b_1 has a neighbor in $B_1 - w_1$, and b_2 has a neighbor in $B_k - w_{k-1}$. When $k = 1$, we set $B_1 - w_1 = B_1 = B_k - w_{k-1}$. Also by planarity and since B^* is $(4, Q + \{x'_1, b_1, b_2\})$-connected, $v_m \in V(B_k - w_{k-1})$.

We may assume that $v_1 \in V(B_1 - w_1)$. This is true when $k = 1$. Now assume $k \geq 2$ and $v_1 \notin V(B_1 - w_1)$. Since B^* is $(4, Q + \{x'_1, b_1, b_2\})$-connected, both x'_1 and b_2 have neighbors in $B_1 - w_1$. By planarity, b_1 has no neighbors in $B_i - w_1$ for all $2 \leq i \leq k$. Let B'' be obtained from the $\{w_1, b_2\}$-bridge of $B^* - x'_1$ containing Q by adding the edge b_2x^* (so that w_1, x', Q, x^*, x^*b_2 occur on $\partial B''$ in order). By Lemma (2.4) we find a $(w_1\partial B''x^*)$-Tutte path P'' from w_1 to b_2 through x^*b_2 and x'. Let B' denote the $\{x'_1, w_1, b_2\}$-bridge of B^* containing b_1. In $B' + \{b_1x'_1, b_2w_1\}$, we apply Lemma (2.4) to find a Tutte path P' from b_1 to b_2 through $b_1x'_1$ and b_2w_1. Now it is easy to check that $P_B := (P' - \{b_1, b_2\}) \cup (P'' - b_2)$ gives the desired path for (2).

For the rest of the proof, we distinguish two cases.

Case 1. b_2 has no neighbor in $B_1 - w_1$.

In this case, $k \geq 2$. Let $x'' \in V(Q)$ with $x'Qx''$ minimal such that x'' is contained in a $(B \cup Q)$-bridge of $B^* - x'_1$ with an attachment not in B_1. Let B'' be the $\{x'', w_1, b_1\}$-bridge of B^* containing $\{x^*, b_2\}$. If all neighbors of b_1 are contained in $B_1 + x'_1$, then we use Lemma (2.4) to find an $x''Qx^*$-Tutte path R in $(B'' - b_1) + \{b_2x^*, x''w_1\}$ from b_2 to w_1 through $x''w_1$ and b_2x^*, and let $P'' := R - \{w_1, b_2\}$. In this case, P'' is in $B'' - \{w_1, b_1, b_2\}$ and from x^* to x'', and $P'' + \{w_1, b_1, b_2\}$ is an $x''Qx^*$-Tutte subgraph of B''. Now assume that b_1 has a neighbor not contained in $B_1 + x'_1$. Note that B'' is $(4, x''Qx^* + \{w_1, b_1, b_2\})$-connected, so we can apply induction to B'' (with w_1, x'' playing the roles of x'_1, x', respectively). Thus either $B'' - \{w_1, b_1, b_2\}$ contains a path P'' from x^* to x'' such that $P'' + \{w_1, b_1, b_2\}$ is an $x''Qx^*$-Tutte subgraph of B'', or $B'' - \{b_1, b_2\}$ has a path P'' from x^* to w_1 through x'' such that $P'' + \{b_1, b_2\}$ is an $x''Qx^*$-Tutte subgraph of B''.

Thus we consider two cases.

Subcase 1.1. The path P'' above is in $B'' - \{w_1, b_1, b_2\}$ and from x^* to x''.

Let B' be obtained from the $\{x'', w_1, b_1\}$-bridge of B^* containing $\{x'_1, x'\}$ by adding $\{b_1w_1, b_2, b_2w_1\}$. Then B' is $(4, x'Qx'' + \{x'_1, b_1, b_2\})$-connected, so we can apply induction to B' (with x'' playing the role of x^*). Hence either $B' - \{x'_1, b_1, b_2\}$ contains a path P' from x'' to x' such that $P' + \{x'_1, b_1, b_2\}$ is an $x'Qx''$-Tutte subgraph of B', or $B' - \{b_1, b_2\}$ contains a path P' from x'' to x'_1 through x' such that $P' + \{b_1, b_2\}$ is an $x'Qx''$-Tutte subgraph of B'. Note that in both cases we must have $w_1 \in V(P')$ since B' is $(4, x'Qx'' + \{x'_1, b_1, b_2\})$-connected and $w_1b_1, w_1b_2 \in E(B')$. Hence, it is easy to check that $P_B := P' \cup P''$ gives the desired path for (1) or (2).

Subcase 1.2. The path P'' above is in $B'' - \{b_1, b_2\}$ and from x^* to w_1 through x''.

Suppose $x' = x''$. Let B' be obtained from the $\{w_1, x''\}$-bridge of $B^* - b_1$ containing $\{x'_1, x'\}$ by adding edges $x''w_1$ and x'_1x', so that $x'_1, x'_1x', x'', x'w_1$ occur on $\partial B'$ in order. We apply Lemma (2.3) to find a $\partial B'$-Tutte path P' from x'_1 to x'' through $x''w_1$. Now $P_B := (P' - x'') \cup P''$ gives the desired path for (2).

Therefore, we may assume $x' \neq x''$. Let $v \in V(x'Qx'' - x'')$ with vQx'' maximal such that v is contained in a $(B \cup Q)$-bridge of $B^* - x'_1$ with an attachment, say w, in $B_1 - w_1$ and such that each $(B \cup Q)$-bridge of $B^* - x'_1$ with an attachment w' in $w\partial B_1 w_1 - \{w, w_1\}$ must be induced by the edge $w'x''$. See Figure 5. Note that v, w are defined since B^* is $(4, Q + \{x'_1, b_1, b_2\})$-connected. In B_1 we apply Lemma (2.3) to find a ∂B_1-Tutte path P_1 from w_1 to w through a vertex which separates the neighbors of b_1 and x'_1 in $\partial B_1 - w_1$. We shall extend $P'' \cup P_1$ to the desired path by finding a path from w to v, and a path from v to x' or x'_1.

Note that v_1 need not be in P_1. Let $J = \{v_1\}$ if $v_1 \in V(P_1)$; otherwise, let J denote the P_1-bridge of B_1 containing v_1 (in which case J has just two attachments on P_1). Let $v' \in V(x'Qv)$ with $v'Qv$ minimal such that v' is contained in a $(B \cup Q)$-bridge of $B^* - x'_1$ with an attachment in $(J - V(P_1 \cap J)) + v_1$.

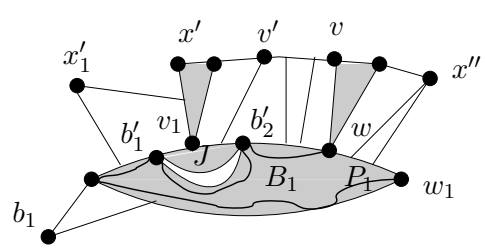


Figure 5: Finding paths Q_1, Q_2, Q_3.

We define a path Q_1 (from v' to $\{x'_1, x'\}$) as follows. If $v_1 \in V(P_1)$, let J_1 denote the union of $x'Qv'$ and those $(B \cup Q)$-bridges of $B^* - x'_1$ whose attachments are all contained

in $x'Qv' + v_1$. Let $J_1' := J_1 + v_1v'$ so that $v_1, x'Qv', v'v_1$ occur on $\partial J_1'$ in order. We apply Lemma (2.3) to J_1' to find a $\partial J_1'$-Tutte path Q_1' from v_1 to x' through v_1v', and let $Q_1 := Q_1' - v_1$. Note that Q_1 is a path from v' to x'. Now assume $v_1 \notin V(P_1)$. Let J^* be the union of J, x_1', and those $((Q + x_1') \cup B)$-bridges of B^* whose attachments are all contained in $(x'Qv' + x_1') \cup (J - V(P_1 \cap J))$. Let b_1', b_2' be the vertices of $P_1 \cap J$ such that b_1', b_2', w_1 occur on ∂B_1 in clockwise order. If x_1' has no neighbor in $J^* - b_1'$, then $x' \neq v'$ (since B^* is $(4, Q + \{x_1', b_1, b_2\})$-connected). In this case, we use Lemma (2.4) to find an $x'Qv'$-Tutte path Q_1' in $(J^* - x_1') + \{b_1'x', b_2'v'\}$ from b_1' to b_2' through $b_1'x'$ and $b_2'v'$, and let $Q_1 := Q_1' - \{b_1', b_2'\}$. Then Q_1 is a path from x' to v' and $Q_1 + \{x_1', b_1', b_2'\}$ is an $x'Qv'$-Tutte subgraph of J^*. Now assume x_1 has a neighbor in $J^* - b_1'$. Note that J^* is $(4, x'Qv' + \{x_1', b_1', b_2'\})$-connected. By applying induction to J^* (with v', b_1', b_2' playing the roles of x^*, b_1, b_2, respectively), we find either a path Q_1 from v' to x' in $J^* - \{x_1', b_1', b_2'\}$ such that $Q_1 + \{x_1', b_1', b_2'\}$ is an $x'Qv'$-Tutte subgraph of J^*, or a path Q_1 from v' to x_1' through x' in $J^* - \{b_1', b_2'\}$ such that $Q_1 + \{b_1', b_2'\}$ is an $x'Qv'$-Tutte subgraph of J^*

Let J_v denote the union of vQx'' and those $(B \cup Q)$-bridges of $B^* - x_1'$ whose attachments are all contained in $vQx'' + w$. In $J_v + wx''$ we apply Lemma (2.3) to find a vQx''-Tutte path Q_2' from x'' to v through $x''w$, and let $Q_2 := Q_2' - x''$. Then Q_2 is a path from w to v.

We apply Lemma (2.5) to find a path Q_3 from v' to v. Let K denote the union of $v'Qv$, B_1, and those $(B \cup Q)$-bridges of $B^* - x_1'$ whose attachments are all contained in $v'Qv \cup B_1$. By Lemma (2.5), with $K, B_1, v'Qv, \partial B_1, P_1, w$ as K, L, Q, Q', T, u, respectively, we find a path Q_3 from v' to v in $K - V(P_1)$ such that $Q_3 \cup P_1$ is a $v'Qv$-Tutte subgraph of K, and any P_1-bridge of B_1 containing no edge of ∂B_1 is a $(Q_3 \cup P_1)$-bridge of K.

Now $P_B := P'' \cup P_1 \cup Q_2 \cup Q_3 \cup Q_1$ is the desired path for (1) or (2).

Case 2. b_2 has a neighbor in $B_1 - w_1$.

First, assume $k \geq 2$. Let $x'' \in V(Q)$ with $x'Qx''$ minimal such that x'' is contained in a $(B \cup Q)$-bridge of $B^* - x_1'$ with an attachment not in B_1. Let B' and B'' denote the $\{x'', w_1, b_2\}$-bridges of B^* containing $\{b_1, x_1', x'\}$ and $\{b_2, x^*\}$, respectively. In $B' + \{b_2w_1, x''w_1\}$, we apply induction (with x'' playing the role of x^*) to find a path P' such that either P' is from x'' to x' in $B' - \{x_1', b_1, b_2\}$ and $P' + \{x_1', b_1, b_2\}$ is an $x'Qx''$-Tutte subgraph of $B' + b_2w_1$, or P' is from x'' to x_1' through x' in $B' - \{b_1, b_2\}$ and $P' + \{b_1, b_2\}$ is an $x'Qx''$-Tutte subgraph of $B' + b_2w_1$. Suppose $x''w_1 \notin E(P')$. Then $w \in V(P')$. In $B'' + \{w_1x'', x^*b_2\}$ we apply Lemma (2.4) to find an $x''Qx^*$-Tutte path P'' from w_1 to b_2 through w_1x'' and x^*b_2. Now $P_B := P' \cup (P'' - \{w_1, b_2\})$ is the desired path for (1) or (2). So we may assume $x''w_1 \in E(P')$. Then in $B'' + b_2x^*$ we apply Lemma (2.4) to find an $x''Qx^*$-Tutte path P'' from w_1 to b_2 through x'' and x^*b_2. Now $P_B := (P' - x'') \cup (P'' - b_2)$ is the desired path for (1) or (2).

So we may assume $k = 1$. Let y denote a vertex in $v_m\partial B_1v_1$ separating the neighbors of b_1 and b_2, and let z denote a vertex in $v_m\partial B_1v_1$ separating the neighbors of b_1 and x_1'.

Let s be maximum such that $s < m$ and v_s is contained in a $(B \cup Q)$-bridge of $B^* - x'_1$ with an attachment in $Q - x^*$.

Suppose $s < m - 1$, or the only $(B \cup Q)$-bridge of $B^* - x'_1$ containing v_m is induced by edge $x^* v_m$. Then let B'_1 be obtained from the subgraph of B^* induced by $V(B_1) \cup \{x^*\}$ by adding the edge $v_s x^*$, such that $\partial B'_1 = v_m \partial B_1 v_s + \{x^*, x^* v_s, x^* v_m\}$. We apply Lemma (2.4) to find an $x^* \partial B'_1 v_s$-Tutte path P_1 in B'_1 from v_s to x^* through y and z. Now by the same argument as for the case when $x' \neq x''$ in Subcase 1.2, with v_s, x^* playing the roles of w, x'' there, we can find paths Q_1, Q_2, Q_3 so that $P_1 \cup Q_1 \cup Q_2 \cup Q_3$ gives the desired path for (1) or (2).

So assume $s = m - 1$ and if $x^* v_m \in E(G)$ then $x^* v_m$ does not induce the unique $(B \cup Q)$-bridge of $B^* - x'_1$ containing v_m. Let P_1 be as in the preceeding paragraph. Let $v \in V(Q)$ with $x'Qv$ minimal such that $\{v, v_s\}$ is contained in a $(B \cup Q)$-bridge of $B^* - x'_1$. Let J_v denote the union of vQx^* and those $(B \cup Q)$-bridges of $B^* - x'_1$ whose attachments are all contained in $vQx^* + \{v_s, v_m\}$. In $J_v + v_s v_m$ we apply Lemma (2.3) to find a vQx^*-Tutte path P_v from v to x^* through $v_s v_m$. By the same argument as in Subcase 1.2 we find paths Q_1 and Q_3 so that $(P_1 - x^*) \cup Q_1 \cup Q_3 \cup (P_v - v_s v_m)$ is the desired path for (1) or (2). □

4 Partial nets

In this section we shall see how tight partial nets may be used to extend Tutte paths. The results discussed here are either from [12] or similar to those in [12]. Let G be a 2-connected infinite plane graph. A *partial net* in G is a sequence $(F_1, \ldots, F_n)$ of vertex disjoint non-dividing cycles in G, where n is a positive integer, such that $I(F_1) = F_1$ and $I(F_i) \subseteq I(F_{i+1})$ for all $1 \leq i \leq n - 1$. If, in addition, for $1 \leq i \leq n - 1$, every $(I(F_i) \cup F_{i+1})$-bridge of $I(F_{i+1})$ has at most one attachment on F_{i+1}, then $(F_1, \ldots, F_n)$ is called a *tight* partial net. The following observation is Lemma (4.1) in [12].

(4.1) Lemma. *Let G be a 2-connected infinite plane graph and let $(F_1, \ldots, F_n)$ be a tight partial net in G. Then the following properties hold.*

(1) There is a plane embedding of G such that $I(F_n)$ is contained in the closed disc in the plane bounded by F_n.

(2) For any distinct $x, y \in V(F_n)$, there is a separation (M_1, M_2) of $I(F_n)$ such that $|V(M_1 \cap M_2)| \leq 2n$, $yF_n x \subseteq M_1$, and $xF_n y \subseteq M_2$.

For a subgraph H of a graph G, we use $N_G(H)$, or $N(H)$ if there is no danger of confusion, to denote the set of vertices in $V(G) - V(H)$ adjacent to vertices of H. We need Lemma (4.2) from [12]. See Figure 6 for an illustration.

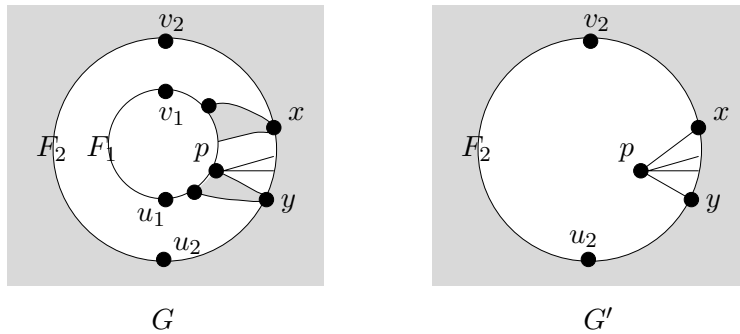


Figure 6: Illustration of Lemma (4.2)

(4.2) Lemma. *Let G be a 2-connected infinite plane graph and (F_1, F_2) be a tight partial net in G such that G is $(4, F_1)$-connected. Suppose $I(F_2)$ is contained in the closed disc bounded by F_2, v_i and u_i are distinct vertices on F_i for $1 \leq i \leq 2$, and $uv \in E(v_1F_1u_1)$ with v_1, v, u, u_1 on $v_1F_1u_1$ in order. Assume there exist two vertex disjoint paths in $I(F_2)$, both from v_1F_1v to $v_2F_2u_2$ or both from uF_1u_1 to $v_2F_2u_2$, which are internally disjoint from $F_1 \cup F_2$. Then there exist $p \in V(v_1F_1u_1 - \{u_1, v_1\})$, distinct $x, y \in V(v_2F_2u_2)$, and $f \in \{px, py\}$ such that*

(1) v_2, x, y, u_2 occur on $v_2F_2u_2$ in order and $N(xF_2y - \{x, y\}) \cap V(I(F_2)) \subseteq \{x, y, p\}$, and

(2) if $G' := (G - V(I(F_2) - (V(F_2) \cup \{p\})) + \{px, py\}$ has a 2-way infinite $(yF_2x + \{p, px, py\})$-Tutte path Q through f then G contains a 2-way infinite F_1-Tutte path P through uv.

Remark. The proof of Lemma (4.2) in [12] actually shows that $Q - p \subseteq P$ and that we only need G to be $(3, F_1)$-connected. Moreover, if there are two disjoint paths from v_1F_1v (respectively, uF_1u_1) to $v_2F_2u_2$, then we may choose $p \in V(v_1F_1v)$ and $f = px$ (respectively, $p \in V(uF_1u_1)$ and $f = py$). Hence, when Lemma (4.2) is applied we allow G to be $(3, F_1)$-connected, and we shall feel free to use the stronger conclusion. The proof of this stronger conclusion is (almost) identical to the proof of the following variation of Lemma (4.2).

Let G be a 2-connected 3-indivisible infinite plane graph and F a facial cycle of G such that F bounds a finite face of G, $G - V(F)$ has a unique infinite component, and G is $(3, F)$-connected. Let H be the minimal union of blocks of $G - V(F)$ containing all infinite blocks of $G - V(F)$, let R be the face of H containing F, and let D denote the maximal subgraph of H in the boundary of R. Note that H is 2-connected or contains two infinite blocks. Assume in the former case, D is a 2-way infinite path (see

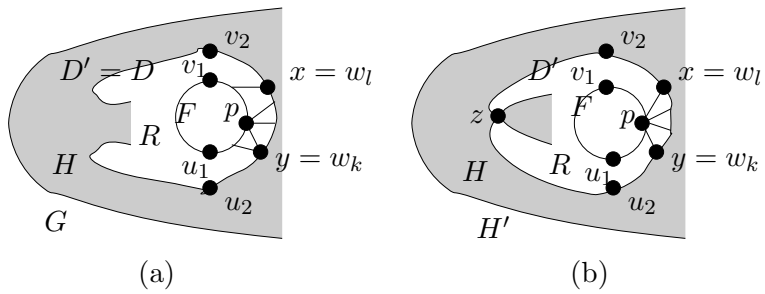


Figure 7: Illustration of Lemma (4.3)

Figure 7(a)), and in the later case, D contains a cycle D' from one infinite block and a 2-way infinite path from the other infinite block (see Figure 7(b)). Let $z \in V(D')$ such that z is a cut vertex of D. For convenience, let $D' := D$ when H is 2-connected. Note that every $(H \cup F)$-bridge of G is finite and has all attachments contained in $F \cup D$, but with at most one on D. Let v_1, u_1 be two distinct vertices of F, let v_2, u_2 be two distinct vertices on D' such that u_1 and u_2 are incident with a common face of G and v_1 and v_2 are incident with a common face of G, and assume $z \notin V(v_2D'u_2)$.

(4.3) Lemma. *Let $G, F, H, R, D, D', z, u_1, v_1, u_2, v_2$ be defined as above. Let uv be an edge of v_1Fu_1 such that v_1, v, u, u_1 occur on v_1Fu_1 in order and there exist two vertex disjoint paths from v_1Fv (respectively, uFu_1) to $v_2D'u_2$ internally disjoint from $F \cup H$. Then there exist $p \in V(v_1Fv)$ (respectively, $p \in V(uFu_1)$) and distinct $x, y \in V(v_2D'u_2)$ such that*

(1) v_2, x, y, u_2 occur on $v_2D'u_2$ in order and $N(xD'y - \{x, y\}) - V(H) \subseteq \{p\}$, and

(2) if $H' := (H + \{p, px, py\}) + \{\text{edges of } G \text{ from } p \text{ to } xD'y - \{x, y\}\}$ has a connected $(D' - V(xD'y - \{x, y\}) + \{p, px, py\})$-Tutte subgraph Q such that $px \in E(Q)$ (respectively, $py \in E(Q)$) and p has degree 2 in Q, then G contains a connected F-Tutte subgraph P containing uv such that $Q - p \subseteq P$, and $P - V(Q - p)$ is a path between the two neighbors of p in Q.

Proof. Let $w_1, \ldots, w_b$ be the attachments on $v_2D'u_2$ of $(H \cup F)$-bridges of G. Without loss of generality, we may assume that $v_2, w_1, \ldots, w_b, u_2$ occur on $v_2D'u_2$ in order. For each w_t, $1 \leq t \leq b$, let $p_t, q_t \in V(F)$ with p_tFq_t maximal such that v_1, p_t, q_t, u_1 occur on F in clockwise order, $\{p_t, w_t\}$ and $\{q_t, w_t\}$ are each contained in an $(H \cup F)$-bridge of G.

We consider only the case when there are two vertex disjoint paths from v_1Fv to $v_2D'u_2$ internally disjoint from $H \cup F$; the other case can be taken care of in the same way. Then $uv \in E(p_kFp_{k+1})$ for some $w_k \in V(v_2D'u_2 - v_2)$. Let J_k denote the union

of p_kFp_{k+1} and those $(H \cup F)$-bridges of G whose attachments are all contained in $p_kFp_{k+1} + w_k$. Then there is some $w_r \in V(v_2D'w_k)$ such that $J_k - p_r$ contains a path from w_k to p_{k+1} through uv. Select w_r so that $w_rD'w_k$ is minimal. Let $w_l = w_r$ if $w_r \neq w_k$; otherwise, let $w_l = w_{k-1}$. Note that $w_l \in V(v_2D'w_k)$. Let J_l denote the union of p_lFp_k and those $(H \cup F)$-bridges of G whose attachments are all contained in $p_lFp_k + w_l$.

Since H is $(3, F)$-connected, we see that all $(H \cup F)$-bridges of G containing some $w_j \in V(w_lF_2w_k - \{w_l, w_k\})$ are induced by the edge p_kw_j. Let $p := p_k$, $x := w_l$, and $y := w_k$. Then $H' := H + \{p_k, p_kw_j : w_j \in V(w_lFw_k)\}$. Let $F' := (D - V(w_lD'w_k - \{w_k, w_l\})) + \{p_k, p_kw_l, p_kw_k\}$.

Assume that H' contains a connected F'-Tutte subgraph Q such that $p_kw_l \in E(Q)$ and p_k has degree 2 in Q. We show next that G has the desired Tutte subgraph P. Note that $Q - p_k$ is a D-Tutte subgraph of H. Moreover, $w_k \in V(Q)$; for otherwise, w_k is contained in a Q-bridge with three attachments (namely, p_k and two on D), a contradiction.

First, let D_1 (respectively, D_2) be the 1-way infinite path in $D - w_l$ (respectively, $D - w_k$) from w_k (respectively, w_l) with R on the same side of D_1 (respectively, D_2). Let $z' \in V(uFv)$ such that every $(H \cup F)$-bridge with an attachment in $D_1 - w_k$ (respectively, $D_2 - w_l$) has all its attachments on F contained in $p_{k+1}Fz'$ (respectively, $z'Fp_l$). Since Q is a connected Tutte subgraph of H', no Q-bridge of H' contains edges from both D_1 and D_2. This enables us to apply Lemma (2.5) to find a path from p_{k+1} to z' and a path from z' to p_l.

Let G_1 denote the union of $p_{k+1}Fz'$, H, and those $(H \cup F)$-bridges of G whose attachments are all contained in $D_1 \cup p_{k+1}Fz'$. By applying Lemma (2.5) to G_1 (with $G_1, H, p_{k+1}Fz', D_1, Q - p_k, w_k$ as K, L, Q, Q', T, u, respectively), we find a path S_1 from p_{k+1} to z' in $G_1 - V(Q - p_k)$ such that $(Q - p_k) \cup S_1$ is a $p_{k+1}Fz'$-Tutte subgraph of G_1, and every $(Q - p_k)$-bridge of H not containing an edge of D_1 is a $((Q - p_k) \cup S_1)$-bridge of G_1.

Let G_2 denote the union of $z'Fp_l$, H, and those $(H \cup F)$-bridges of G whose attachments are all contained in $D_2 \cup z'Fp_l$. By applying Lemma (2.5) to G_2 (with $G_2, H, z'Fp_l, D_2, Q - p_k, w_l$ as K, L, Q, Q', T, u, respectively), we find a path S_2 from p_l to z' in $G_2 - V(Q - p_k)$ such that $(Q - p_k) \cup S_2$ is a $z'Fp_l$-Tutte subgraph of G_2, and every $(Q - p_k)$-bridge of H not containing an edge of D_2 is a $((Q - p_k) \cup S_2)$-bridge of G_2.

Since p_k has degree 2 in Q, let $p_kw_j \in E(Q)$ with $w_j \neq w_l$. We shall complete $(Q - p_k) \cup S_1 \cup S_2$ to the desired path in G by finding a path from w_j to p_{k+1} and a path from w_l to p_l.

Suppose $p_k \neq p_l$. In $J_l + w_lp_k$ we apply Lemma (2.3) to find a p_lFp_k-Tutte path P'_l from p_l to p_k such that $w_lp_k \in E(P'_l)$, and let $P_l := P'_l - p_k$. If $w_k = w_j$, then in $J_k + w_kp_{k+1}$ we apply Lemma (2.4) to find a p_kFp_{k+1}-Tutte path P_k from w_k to p_{k+1}

through p_k and uv. If $w_k \neq w_j$, then in $J_k + \{w_k p_k, w_k p_{k+1}\}$ we apply Lemma (2.4) to find a $p_k F p_{k+1}$-Tutte path P'_k from w_k to p_{k+1} through $w_k p_k$ and uv; and let $P_k := (P'_k - w_k) + \{w_j, p_k w_j\}$. Let $P := (Q - p_k) \cup P_k \cup S_1 \cup S_2 \cup P_l$. It is easy to verify that P is the desired F-Tutte subgraph of G.

Now assume $p_k = p_l$. Then $w_l = w_{k-1}$, $w_j = w_k$, J_l is induced by the edge $w_l p_l$, and $J_k - p_k = J_k - p_l$. Since $J_k - p_l$ has a path T from w_k to p_{k+1} through uv, we let J'_k denote the union of blocks of $J_k - p_l$ each of which contains an edge of T. Let T' denote the path from w_k to p_{k+1} containing uv such that T' is on the boundary of the face of $J_k - p_l$ containing R. Let $p' \in V(F \cap J'_k)$ with $p_k F p'$ minimal. By applying Lemma (2.4) we find a T'-Tutte path P_k in $J'_k + w_k p_{k+1}$ from w_k to p_{k+1} through p' and uv. Let $P := ((Q - p_k) \cup P_k \cup S_1 \cup S_2) + w_l p_l$. It is easy to verify that P is the desired F-Tutte subgraph of G. □

Remark. If Q in Lemma (4.3) is a 2-way infinite path then so is P.

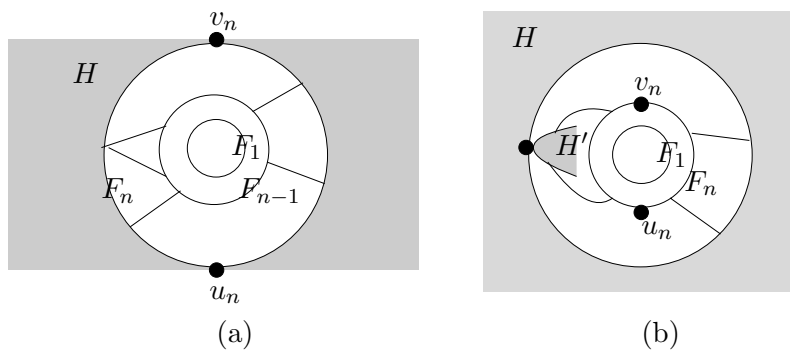


Figure 8: Illustration of Lemma (4.4)

The next lemma is Lemma (4.3) in [12]. Let G be a 2-connected infinite plane graph which is 3-indivisible but not 2-indivisible, let $(F_1, \ldots, F_n)$ be a tight partial net in G such that G is $(4, F_1)$-connected, and assume that $I(F_n)$ is drawn in the closed disc bounded by F_n. For $1 \leq i \leq n$, let u_i, v_i be distinct vertices of F_i such that any two consecutive vertices from $u_n, \ldots, u_1, v_1, \ldots, v_n$ are contained in a facial cycle of $I(F_n)$ (and hence, of G), and assume that there is no separation (H_1, H_2) of $I(F_n)$ such that $|V(H_1 \cap H_2)| < 2n$, $\{u_n, v_n\} \subseteq V(H_1 \cap H_2)$, $v_n F_n u_n \subseteq H_1$, and $u_n F_n v_n \subseteq H_2$.

(4.4) Lemma. *Let $G, (F_1, \ldots, F_n)$, and u_i, v_i, $1 \leq i \leq n$, be defined as above. Suppose either (1) $(G - V(I(F_n) - V(F_n))) - \{u_n, v_n\}$ has two infinite components, or (2) $G - V(I(F_n))$ has two infinite blocks, say H and H', such that the face of H containing $I(F_n)$ contains $H' - V(H \cap H')$ and is bounded by a cycle, and such that no path in G from H' to $v_n F_n u_n - \{v_n, u_n\}$ is internally disjoint from $I(F_n) \cup H$. Let $uv \in E(v_1 F_1 u_1)$ such*

that v_1, v, u, u_1 occur on $v_1F_1u_1$ in order, and assume when $n \geq 2$ there exist two vertex disjoint paths in $I(F_2)$, both from v_1F_1v to $v_2F_2u_2$ or both from u_1F_1u to $v_2F_2u_2$, which are internally disjoint from $F_1 \cup F_2$. Then there is a 2-way infinite F_1-Tutte path in G through uv.

5 Structure

We now start with the proof of Theorem (1.2). Let G be a 4-connected 3-indivisible infinite plane graph such that $\gamma(G)$ is a positive integer. We first use a dividing cycle C in G to divide the graph G into two infinite plane graphs G' and G'' whose intersction is C. Now both G' and G'' are 2-indivisible graphs, and therefore we may use Theorem (2.2) to describe their structures in terms of ladder nets. We then pick a curve in the plane to define a separation (K', K'') of G, and show that (K', K'') may be chosen to minimize K' (in terms of subgraph containment). Using (K', K'') we construct a partial tight net $\mathcal{F}$ which, then, is used to construct a sequence $(H_i, R_i, x_i, y_i, D_i)$ for $1 \leq i \leq q$, where H_i is an infinite plane graph, R_i is a 2-way infnite path in H_i, x_i and y_i are distinct vertices on R_i, and $D_i = x_iR_iy_i$. This sequence will be used in the next section to reduce our proof of Theorem (1.2) to a Tutte path problem in some infinite plane graph.

First, we use Theorem (2.2) to derive structural information about G.

(5.1) *Structure of G.* Let C be a dividing cycle in G, let G'' be the subgraph of G contained in the closed disc of the plane bounded by C, and let $G' := G-(V(G'')-V(C))$. (Note that G' is the subgraph of G contained in the other closed region bounded by C.) Then $G' \cap G'' = C$, and both G' and G'' are 2-indivisible and $(4, C)$-connected.

Let S' denote the set of vertices of G' of infinite degree. By Theorem (2.2), there is a set $F' \subseteq E(G')$ whose elements are incident with vertices in S' such that $G' - F'$ has a net $N' = (C'_1, C'_2, \ldots)$ satisfying the conclusions of Theorem (2.2) (with G', S', F', N' as G, S, F, N, respectively).

Similarly, let S'' denote the set of vertices of G'' of infinite degree. By Theorem (2.2), there is a set $F'' \subseteq E(G'')$ whose elements are incident with vertices in S'' such that $G'' - F''$ has a net $N'' = (C''_1, C''_2, \ldots)$ satisfying the conclusions of Theorem (2.2) (with G'', S'', F'', N'' as G, S, F, N, respectively).

Since $\gamma(G)$ is a positive integer, G does not contain infinitely many vertex disjoint dividing cycles. So both N' and N'' are ladder nets. Note that $\partial G' \subseteq \partial N'$ and $\partial G'' \subseteq \partial N''$.

(5.2) *Description of a curve β and a separation (K', K'').* There exist vertices x, y in G (not necessarily distinct) and a simple curve β in the plane between x and y such that

(i) $x, y \in V(\partial G')$ or $x, y \in V(\partial G'')$, $\beta \cap G \subseteq V(G)$, and $G - (\beta \cap G)$ has two infinite components,

(ii) subject to (i), $|\beta \cap G|$ is minimum, and

(iii) subject to (ii) and, if possible, $x \neq y$.

For any two points a, b on β, we use $a\beta b$ to denote the subcurve of β between a and b. (In the case of $x = y$, it will be clear from context which subcurve of β that $a\beta b$ refers to.) Without loss of generality, we assume that $\{x, y\} \subseteq V(\partial G')$, and let the notation of x and y be chosen so that

(iv) $x\partial G'y = xC'_j y$ for all j.

Let (K', K'') be the separation of G such that $V(K' \cap K'') = \beta \cap G$, and if $x \neq y$ (respectively, $x = y$) $yC'_j x \subseteq K'$ (respectively, $C'_j \subseteq K'$) for all sufficiently large j.

(5.3) We may assume that there are only finitely many separations (L', L'') such that $\{x, y\} \subseteq V(L' \cap L'')$, $|V(L' \cap L'')| = |V(K' \cap K'')|$, $K'' \subseteq L''$, and L' contains C'_j for all sufficiently large j.

Proof. When $|(\beta \cap G) - \{x, y\}|$ is even, let $x = u_m, u_{m-1}, \cdots, u_1, v_1, \cdots, v_{m-1}, v_m = y$ be the vertices of $L' \cap L''$ that occur on β in order, and when $|(\beta \cap G) - \{x, y\}|$ is odd, let $x = u_m, u_{m-1}, \cdots, u_1, w, v_1, \cdots, v_{m-1}, v_m = y$ be the vertices of $L' \cap L''$ that occur on β in order.

Suppose (5.3) is not true. Then we may choose (L', L'') so that $V(L' \cap L'')$ is disjoint from $I_{G'}(C'_{m+2}) - \{x, y\}$. Next we show that G has a spanning 2-way infinite path, and hence Theorem (1.2) holds.

Let $G^* := G$ if $|(\beta \cap G) - \{x, y\}|$ is even, and let A_1 be the facial cycle of G^* incident with both u_1 and v_1. When $|(\beta \cap G) - \{x, y\}|$ is odd, let A be the facial cycle of $G - w$ incident with both u_1 and v_1, let w_1, w_2 be two distinct neighbors of w in $v_1 A u_1 - \{v_1, u_1\}$ such that v_1, w_1, w_2, u_1 occur on A in order and w has no neighbors in $w_1 A w_2 - \{w_1, w_2\}$, and let $A_1 := w_2 A w_1 + w_1 w_2$ and $G^* := (G - w) + w_1 w_2$. Note that the cycles A, A_1 exist by Lemma (2.1), because $u_1, v_1 \notin V(\partial G' \cup \partial G'')$. Also note that $\{u_1, v_1\}$ is disjoint from $I_{G'}(C'_{m+1})$.

We construct a tight partial net in G^* as follows. Suppose we have constructed a tight partial net $(F_1, \ldots, F_i)$ in G^* for some $1 \leq i < m$ such that $\{u_j, v_j\} \subseteq V(F_j)$ for $1 \leq j \leq i$, and $\{u_j, v_j : i+1 \leq j \leq m\}$ is disjoint from $I_{G^*}(F_i)$. Since $V(L' \cap L'') - \{x, y\}$ is disjoint from $I_{G'}(C'_{m+2})$ and by the minimality of $|\beta \cap G| = |V(L' \cap L'')|$ in (5.2)(ii) (and by Lemma (2.1)), we see that $G^* - V(I_{G^*}(F_i))$ has a unique infinite block whose face containing $I_{G^*}(F_i)$ is bounded by a cycle F_{i+1}. Clearly, $(F_1, \ldots, F_{i+1})$ is a tight partial net in G^*, $\{u_{i+1}, v_{i+1}\} \subseteq V(F_{i+1})$, and $\{u_j, v_j : i+2 \leq j \leq m\}$ is disjoint from $I_{G^*}(F_{i+1})$.

If $x \neq y$, then the above process produces a tight partial net $(F_1, \ldots, F_m)$ in G^* such that $(G^* - V(I_{G^*}(F_m) - V(F_m))) - \{x, y\}$ has two infinite components. If $x = y$

then the above process produces a tight partial net $(F_1, \ldots, F_{m-1})$ in G^* such that $G^* - V(I_{G^*}(F_{m-1}))$ has two infinite blocks. Therefore, (1) or (2) of Lemma (4.4) holds. It is easy to verify that other conditions of Lemma (4.4) hold for some edge $uv \in E(v_1F_1u_1)$, with $uv = w_1w_2$ when $|(\beta \cap G) - \{x, y\}|$ is odd. Thus, by Lemma (4.4), G^* contains a 2-way infinite F_1-Tutte path P through uv. Since G is 4-connected, P or $(P - w_1w_2) + \{w, ww_1, ww_2\}$ is a spanning 2-way infinite path in G. □

By (5.3) above, we may choose β and (K', K'') so that

(5.4) K' is minimal (with respect to subgraph containment).

(5.5) *Description of H, R, D and vertices $x = u_m, \ldots, u_1, (w), v_1, \ldots, v_m = y$.*

When $|(\beta \cap G) - \{x, y\}|$ is even, let $x = u_m, u_{m-1}, \cdots, u_1, v_1, \cdots, v_{m-1}, v_m = y$ be the vertices of G that occur on β in order, let $H := G - F''$, let R be the face of H whose boundary is $\partial N''$, and let $D := \partial N''$. When $|(\beta \cap G) - \{x, y\}|$ is odd, let $x = u_m, u_{m-1}, \cdots, u_1, w, v_1, \cdots, v_{m-1}, v_m = y$ be the vertices of G that occur on β in order, let $H := (G - w) - F''$, and let R denote the face of H whose boundary is a 2-way infinite path D containing $\partial N'' - w$. See Figure 9 (with or without w).

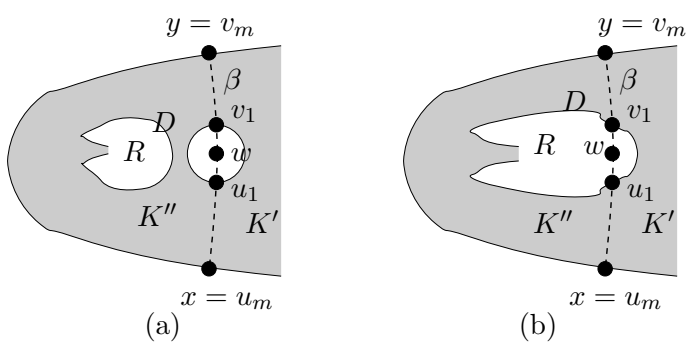


Figure 9: The graph H, face R, and 2-way infinite path D

Since G is 4-connected, $m \geq 2$. We now construct a (possibly empty) tight partial net $\mathcal{F} := (F_1, \ldots, F_n)$ for some $0 \leq n \leq m$.

(5.6) *Construction of $\mathcal{F}$.* If R is the face of H that is incident to both u_1 and v_1 and intersects β, then $\{u_1, v_1\} \subseteq V(D)$ and $v_1Du_1 \subseteq K'$ (by the minimality of K' in (5.4)), and let $\mathcal{F} = \emptyset$. See Figure 9(b).

Now assume that R is not the face of H that is incident with both u_1 and v_1 and intersects D. Since G is 4-connected and $u_1, v_1 \notin V(\partial G')$ and by Lemma (2.1), the face of H intersecting β and incident with both u_1 and v_1 is bounded by a cycle, say F_1. See Figure 9(a). By the minimality of K' in (5.4), $v_1F_1u_1 \neq v_1u_1$ and $v_1F_1u_1 \subseteq K'$.

Suppose that we have constructed in H a tight partial net $(F_1, \ldots, F_l)$ for some integer $l \geq 1$. If $l = m$ (Figure 10(a)), or $H - V(I_H(F_l))$ contains two infinite blocks (Figure 10(b)), or $F_l \cap D \neq \emptyset$ (Figure 10(c)), then we stop this construction, and let $n := l$ and $\mathcal{F} := (F_1, \ldots, F_l)$.

Otherwise, $l < m$, and $H - V(I_H(F_l))$ has a unique infinite block whose face containing $I_H(F_l)$ is bounded by a cycle, say F_{l+1}. By the minimality of $|\beta \cap G|$ in (5.2)(ii) and by applying Lemma (4.1), $\{u_{l+1}, v_{l+1}\} \subseteq V(F_{l+1})$, and each $(I_H(F_l) \cup F_{l+1})$-bridge of $I_H(F_{l+1})$ has at most one attachment on F_{l+1}. So $(F_1, \ldots, F_l, F_{l+1})$ is a tight partial net in H.

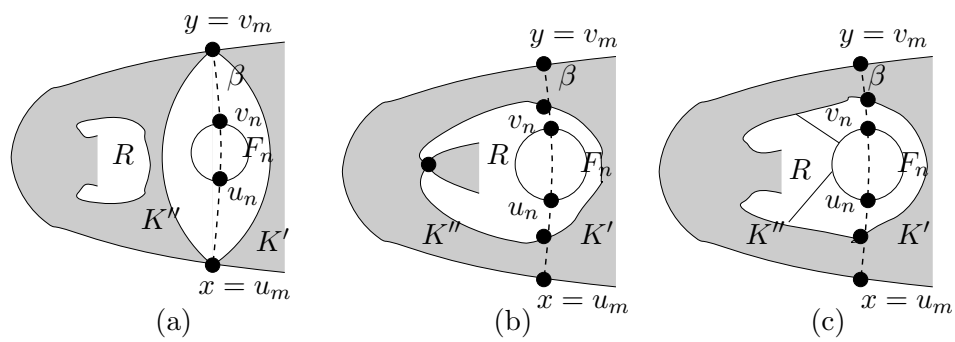


Figure 10: Tight partial net $\mathcal{F}$.

Let $\mathcal{F} := (F_1, \ldots, F_n)$ be the maximal tight partial net constructed in this way. If $n = m$ (Figure 10(a)) then $\mathcal{F}$ is a tight partial net in G and $(G - V(I(F_n) - F_n)) - \{u_n, v_n\}$ has two infinite components (condition (1) of Lemma (4.4)). If $H - V(I_H(F_n))$ has two infinite blocks (Figure 10(b)), then we can verify that condition (2) of Lemma (4.4) holds. Therefore, if w is not defined, then by Lemma (4.4) and since G is 4-connected, G has a spanning 2-way infinite path. Now assume w is defined. Then by the minimality of K', w has two neighbors w_1, w_2 in $v_1F_1u_1 - \{v_1, u_1\}$ such that v_1, w_1, w_2, u_1 occur on F_1 in order and w has no neighbor in $w_1F_1w_2 - \{w_1, w_2\}$. Let $F_1' := w_2F_1w_1 + w_1w_2$. It is easy to verify that $H + w_1w_2$ has a tight partial net $(F_1', F_2', \ldots, F_n')$ such that $F_i' \cap K'' = F_i \cap K''$. Note that conditions (1) or (2) of Lemma (4.4) holds for $(F_1', \ldots, F_n')$. So by Lemma (4.4), $H + w_1w_2$ has a 2-way infinite F_1'-Tutte path P through w_1w_2. Since G is 4-connected, $(P - w_1w_2) + \{w, ww_1, ww_2\}$ is a spanning 2-way infinite path in G. So we may assume

(5.7) $n < m$, and $H - V(I_H(F_n))$ has a unique infinite block.

Hence, if $\mathcal{F} \neq \emptyset$ then $D = \partial N''$ and $D \cap F_n \neq \emptyset$. See Figure 10(c). By the choice of β in (5.2) and by Lemma (4.1), it is easy to see that $\{u_i, v_i\} \subseteq V(F_i)$ for $1 \leq i \leq n$.

(5.8) *Construction of* $(H_1, R_1, x_1, y_1, D_1)$. We proceed by considering two cases.

Case 1. If $\mathcal{F} \neq \emptyset$ and $|S'' \cap V(F_n)| = 2$, then let $x_1, y_1 \in S''$ such that u_n, x_1, y_1, v_n occur on F_n in order, and let $D_1 := y_1 F_n x_1$. See Figure 11(a). Now assume $\mathcal{F} \neq \emptyset$ and $S'' \cap V(F_n)$ has exactly one vertex, say z, with infinitely many neighbors in both z-bridges of $\partial N''$. By the minimality of $|\beta \cap G|$ in (5.2)(ii), there exists a path from some $z' \in V(u_n F_n v_n - z)$ to $D - v_{n+1} D u_{n+1}$ internally disjoint from $D \cup F_n$. If $z' \in V(z F_n v_n)$ then choose z' so that $z F_n z'$ is minimal and let $x_1 = z$ and $y_1 = z'$; otherwise, choose z' so that $z' F_n z$ is minimal and let $x_1 = z'$ and $y_1 = z$. See Figure 11(a). Let $D_1 := y_1 F_n x_1$.

Let H_1 be the $\{x_1, y_1\}$-bridge of $H - V(I_H(F_n) - V(F_n))$ containing D_1, and let R_1 denote the 2-way infinite path in the boundary of the face of H_1 containing R. Then $D_1 = y_1 R_1 x_1$. For convenience, let $u'_i := u_{i+n-1}$ and $v'_i := v_{i+n-1}$ for $1 \leq i \leq m-(n-1)$.

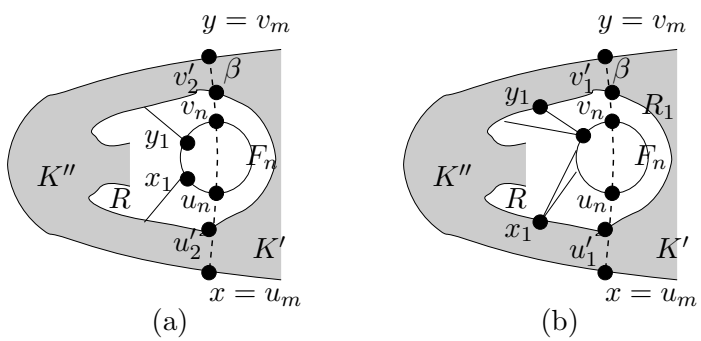


Figure 11: R_1, x_1, y_1 and D_1.

Case 2. $\mathcal{F} = \emptyset$, or $S'' \cap V(F_n) = \emptyset$, or $S'' \cap V(F_n)$ consists of one vertex and only one $(S'' \cap V(F_n))$-bridge of $\partial N''$ contains infinitely many neighbors of $S'' \cap V(F_n)$.

If $\mathcal{F} = \emptyset$ let $H_1 := H$ and $R_1 := D$. If $\mathcal{F} \neq \emptyset$, then by (5.7) let H_1 be the unique infinite block of $H - V(I_H(F_n))$, and let R_1 denote the 2-way infinite path in the boundary of the face of H_1 containing R. See Figure 11(b).

We now define x_1, y_1, D_1. By the minimality of $|\beta \cap G|$ in (5.2)(ii), $\{u_{n+1}, v_{n+1}\} \subseteq V(R_1)$. Let x_1 (respectively, y_1) be the vertex in the $u_{n+1} R_1 v_{n+1}$-bridge of R_1 containing u_{n+1} (respectively, v_{n+1}) with one of the following properties:

(1) $x_1 \in S''$ (respectively $y_1 \in S''$); or

(2) x_1 (respectively, y_1) is a neighbor of the unique vertex in $S'' \cap V(F_n)$; or

(3) $x_1 = u_{n+1}$ (respectively, $y_1 = v_{n+1}$) or x_1 (respectively, y_1) is contained in an $(H_1 \cup I_H(F_n))$-bridge of H, and subject to this, $x_1 R_1 u_{n+1}$ (respectively, $y_1 R_1 v_{n+1}$) is maximal.

Let $D_1 := y_1 R_1 x_1$. See Figure 11(b). Let $u'_i := u_{i+n}$ and $v'_i := v_{i+n}$ for $1 \leq i \leq m-n$.

(5.9) *Construction of* $(H_i, R_i, x_i, y_i, D_i)$ *for* $1 \le i \le q$.

Suppose for some $k \ge 1$ we have constructed $(H_i, R_i, x_i, y_i, D_i)$, $1 \le i \le k$, such that for $2 \le i \le k$, H_i is the unique infinite block of $H_{i-1} - V(D_{i-1})$, R_i is the 2-way infinite path in the boundary of the face of H_i containing R, $\{u'_i, v'_i\} \subseteq V(R_i)$, x_i (respectively, y_i) is contained in the $u'_i R_i v'_i$-bridge of R_i containing u'_i (respectively, v'_i), and $D_i = y_i R_i x_i$.

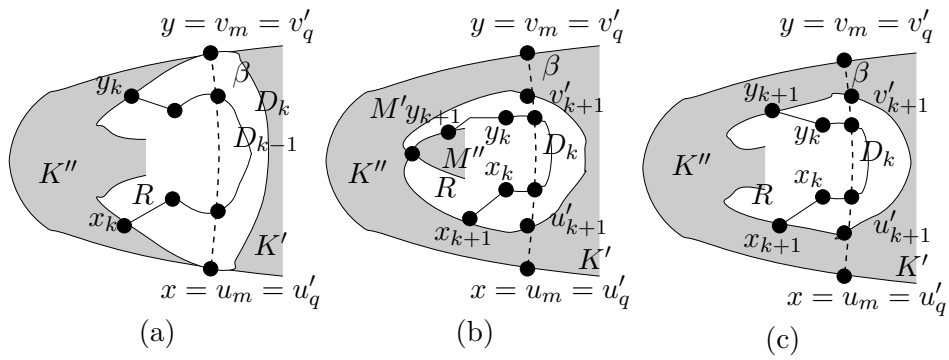


Figure 12: Construction of H_k.

If $\{u'_k, v'_k\} = \{u_m, v_m\}$, we stop, and let $q := k$. See Figure 12(a). Now assume $\{u'_k, v'_k\} \neq \{u_m, v_m\}$.

Suppose $H_k - V(D_k)$ has two infinite blocks, say M' and M'', with M' containing C'_j for all j. See Figure 12(b). Let H_{k+1} denote the minimal union of blocks of $H_k - V(D_k)$ containing M' and M''. Let R_{k+1} denote the maximal subgraph of H_{k+1} in the boundary of the face of H_{k+1} containing R. Note that R_{k+1} contains $\partial M''$ and a facial cycle of M'. If $k = 1$ and $x_1 \in S''$ (respectively, $y_1 \in S''$), we let x_{k+1} (respectively, y_{k+1}) be a neighbor of x_k (respectively, y_k) in $M'' \cap R_{k+1}$. Otherwise, let $x_{k+1} \in V(R_{k+1})$ (respectively, $y_{k+1} \in V(R_{k+1})$) such that x_{k+1} (respectively, y_{k+1}) is incident with R, and $\{x_k, x_{k+1}\}$ (respectively, $\{y_k, y_{k+1}\}$) is in a $(D_k \cup H_{k+1})$-bridge of H_k. Let $q := k+1$ and $D_{k+1} = \emptyset$, and we stop this process.

Now suppose $H_k - V(D_k)$ has a unique infinite block, say H_{k+1}. See Figure 12(c). Let R_{k+1} denote the 2-way infinite path in the boundary of the face of H_{k+1} containing R. If $k = 1$ and $x_1 \in S''$ (respectively, $y_1 \in S''$) then let x_{k+1} (respectively, y_{k+1}) be a neighbor of x_1 (respectively, y_1) in the $v'_{k+1} R_{k+1} u'_{k+1}$-bridge of R_{k+1} containing u'_{k+1} (respectively, v'_{k+1}). Otherwise, let $x_{k+1} \in V(R_{k+1})$ (respectively, $y_{k+1} \in V(R_{k+1})$) such that $\{x_{k+1}, x_k\}$ (respectively, $\{y_k, y_{k+1}\}$) is contained in an $(H_{k+1} \cup D_k)$-bridge of H_k such that $x_k R_{k+1} x_{k+1}$ (respectively, $y_k R_{k+1} y_{k+1}$) is maximal. Let $D_{k+1} := x_{k+1} R_{k+1} y_{k+1}$.

If $\{u'_{k+1}, v'_{k+1}\} = \{u_m, v_m\}$, then we stop; otherwise, we repeat this process. Let $(H_i, R_i, x_i, y_i, D_i), 1 \le i \le q$, be the maximal sequence constructed in this way. Thus H_q has two infinite blocks, and H_i, $1 \le i \le q-1$, are 2-connected. Note that for

$1 \leq i \leq q-1$, every $(H_{i+1} \cup D_i)$-bridge of H_i has at most one attachment on H_{i+1}.

Without loss of generality (by symmetry between x_1 and y_1), we may assume that

(5.10) if $S'' \neq \emptyset$ then either $y_1 \in S''$, or $\mathcal{F} = \emptyset$, $w \in S''$, and $y_1 w \in E(G)$.

We note here that in our later reductions, $H_i \cap K''$ will remain unchanged.

6 Reduction

The goal of this section is to use the sequence $(H_i, R_i, x_i, y_i, D_i)$ to reduce the proof of Theorem (1.2) to a Tutte path problem. Our reduction depends on the existence of certain Tutte subgraphs in $H_q \cap K''$, which we analyze first.

(6.1) *Descriptions of M, B and P_M in $H_q \cap K''$.* This will be done in two cases according to whether or not H_q is 2-connected; and in each case, we consider two subcases according to the parity of q.

Case 1. H_q has two infinite blocks. See Figure 13(a).

Let M' and M'' be the infinite blocks of H_q such that $C'_j \subseteq M'$ for all large j and $M'' \subseteq K''$. Let v' (respectively, v'') be the cut vertex of H_q contained in M' (respectively, M''). Let M denote the v'-bridge of H_q containing M''.

We must have $\{x_q, y_q\} \cap V(M'' - v'') \neq \emptyset$; for, otherwise, $S'' \cup \{v''\}$ is a cut set in G (see (5.8) and (5.9)), a contradiction, since G is 4-connected and $|S''| \leq 2$. In fact, we must have $\{x_q, y_q\} \subseteq V(M'')$. For, otherwise, it follows from definitions in (5.8) and (5.9) that, unless $q = 1$ and $x = y = v'$, we would have a simple curve β' between two vertices of $\partial G''$ through v'' such that $|\beta' \cap G| < |\beta \cap G|$ and $G - (\beta' \cap G)$ has two infinite components. In the case where $q = 1$ and $x = y = v'$, we see from the definition of x_q, y_q in (5.8) that we may choose β between two distinct vertices of $\partial G''$ (one of which is v''), contradicting the choice of β in (5.2)(iii).

Subcase 1.1. q is odd.

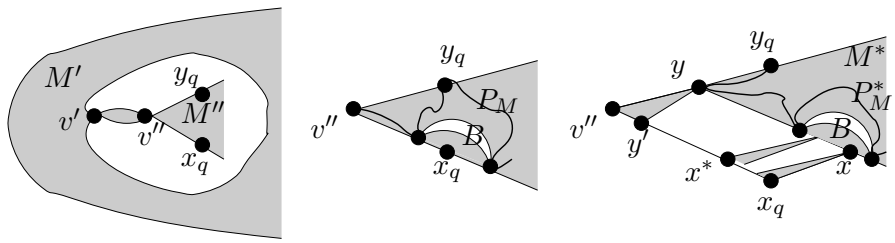
By applying Lemma (2.6) to M'' (and Lemma (2.3) to other blocks of M), we find a 1-way infinite ∂M-Tutte path P_M in M from v' through y_q. We require $y_q \in V(P_M)$ because of (5.10).

If $x_q \in V(P_M)$ let $B = \{x_q\}$, and if $x_q \notin V(P_M)$ let B denote the P_M-bridge of M containing x_q (which has exactly two attachments on P_M). See Figure 13(b).

Subcase 1.2. q is even.

Then $q \geq 2$. Hence $\{x_q, y_q\} \subseteq V(M'' - v'')$. For, otherwise, there is a simple curve β' between vertices of $\partial G''$ such that $G - (\beta' \cap G)$ has two infinite components and $|\beta' \cap G| < |\beta \cap G|$, a contradiction.

Let M^* denote the minimal union of blocks of $M'' - V(x_q \partial M'' v'')$ containing a 1-way infinite path from y_q. Let $y \in V(M^* \cap y_q \partial M'' v'')$ with $y \partial M'' v''$ minimal, and let



(a) (b) q is odd. (c) q is even.

Figure 13: Construction of P_M and B in Case 1.

$x \in V(M^* \cap \partial M'')$ such that $\{x, x_q\}$ is contained in an $(M^* \cup x_q \partial M'' v'')$-bridge of M''. See Figure 13(c).

Note that M^* has a finitely many blocks and exactly one of which is infinite. Moreover, the infinite block of M^* contains a 1-way infinite subpath of $\partial M^* \cap \partial M''$. By applying Lemma (2.6) to the infinite block of M^* (and Lemma (2.3) to finite blocks of M^*), we find a 1-way infinite ∂M^*-Tutte path P_M^* from y_q through y.

Let $B = \{x\}$ if $x \in V(P_M^*)$; otherwise, let B denote the P_M^*-bridge of M^* containing x. Let $x^* \in V(x_q \partial M'' v'')$ with $x^* \partial M'' v''$ minimal such that x^* is contained in an $(M^* \cup x_q \partial M'' v'')$-bridge of M'' with an attachment in $B - V(P_M^* \cap B)$. Next, we complete P_M^* to P_M by finding a path from v'' to y', a path from y' to x^*, and a path from v'' to v', where $y' \in V(x_q \partial M'' v'')$ with $y' \partial M'' v''$ maximal such that $\{y', y\}$ is contained in an $(M^* \cup x_q \partial M'' v'')$-bridge of M''.

Let J_y denote the union of $v'' \partial M'' y'$ and those $(M^* \cup x_q \partial M'' v'')$-bridges of M'' whose attachments are all contained in $v'' \partial M'' y' + y$. In $J_y + yy'$ we apply Lemma (2.3) to find a $y \partial M'' y'$-Tutte path P_v from v'' to y through yy'.

Let K be the union of M^*, $x^* \partial M'' y'$, and those $(M^* \cup x_q \partial M'' v'')$-bridges of M'' whose attachments are all contained in $x^* \partial M'' y' \cup M''$. By applying Lemma (2.5) (with $K, M^*, x^* \partial M'' y', \partial M^*, P_M^*, y$ as K, L, Q, Q', T, u, respectively), we find a path P_y' in $K - V(P_M^*)$ from y' to x^* such that $P_y' \cup P_M^*$ is an $x^* \partial M'' y'$-Tutte subgraph of K and every P_M^*-bridge of M^* containing no edge of ∂M^* is also a $(P_y' \cup P_M^*)$-bridge of K.

By applying Lemma (2.3) to the blocks of M other than M'' (possibly none), we find a $(\partial M - (\partial M'' - v''))$-Tutte path P_v' form v'' to v'.

Then $P_x := (P_v - y) \cup P_y' \cup P_v'$ is a path from x^* to v', and $P_x \cap P_M^* = \emptyset$. Let $P_M := P_x \cup P_M^*$. Then P_M is a ∂M-Tutte subgraph of M, with the exception of the P_M-bridge containing $B \cup x_q \partial M'' x^*$, which has three attachments, x^* and two in $V(P_M^* \cap B)$.

Case 2. H_q is 2-connected.

Then $\{u_q', v_q'\} = \{u_m, v_m\}$. See Figure 14. Let $M := H_q \cap K''$. Let $u_q'', v_q'' \in V(u_q' \partial G' v_q')$ with $u_q'' \partial G' v_q''$ maximal such that u_q', u_q'', v_q'', v_q' occur on $\partial G'$ in order, and

the $\{u''_q, v''_q\}$-bridge of M, say M', containing $u''_q \partial G' v''_q$ is 2-connected.

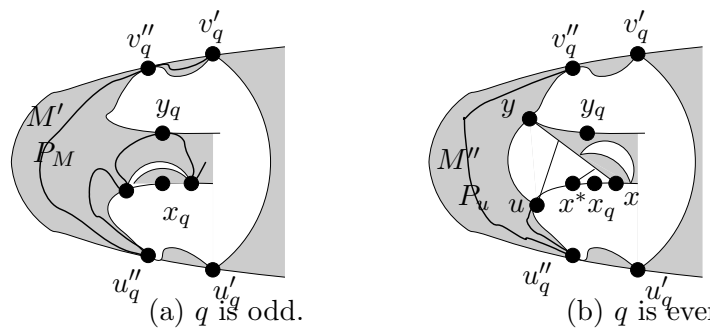
Subcase 2.1. q is odd.

If $y_q \notin V(M')$ then by Lemma (2.6) we find a 1-way infinite $\partial M'$-Tutte path P'_M from v''_q through u''_q. Otherwise, by Lemma (3.1), M' contains a 1-way infinite $\partial M'$-Tutte path P'_M from $\{u''_q, v''_q\}$ through y_q. See Figure 14(a).

If P'_M is from v''_q, then we can find a path P''_M as follows. If $v'_q = v''_q$ then let $P''_M = \emptyset$; otherwise, in the v''_q-bridge of M containing v'_q, we apply Lemma (2.3) (to its blocks) to find a $v'_q R_q v''_q$-Tutte path P''_M from v'_q to v''_q through y_q (if $y_q \notin V(M')$). If P'_M is from u''_q, then we can find a path P''_M from u'_q to u''_q through x_q (if $x_q \notin V(M')$) in the same way.

Let $P_M = P'_M \cup P''_M$. Then P_M is a 1-way infinite $(R_q \cap M)$-Tutte path in M from $\{u'_q, v'_q\}$ through y_q.

Let $B = \{x_q\}$ if $x_q \in V(P_M)$; and otherwise, let B be the P_M-bridge of M containing x_q. Note that B has at most two attachments on P_M; and in the case it has just one attachment, B is the u''_q-bridge of M containing u'_q.



(a) q is odd. (b) q is even.

Figure 14: Construction of P_M and B in Case 2.

Subcase 2.2. q is even.

Let M^* denote the minimal union of blocks of $M - V(x_q R_q u'_q)$ containing a 1-way infinite path from y_q. Let $y \in V(M^* \cap v'_q R_q y_q)$ with $y R_q v'_q$ minimal. Let $x \in V(M^* \cap R_q)$ such that $x_q \in V(u'_q R_q x)$ and $\{x, x_q\}$ is contained in some $(M^* \cup x_q R_q u'_q)$-bridge of M. See Figure 14(b). Note that M^* has finitely many blocks exactly one of which is infinite.

In M^*, we use Lemma (2.6) (and Lemma (2.3) for the finite blocks of M^*) to find a 1-way infinite ∂M^*-Tutte path P_y from y_q through y. Let $B = \{x\}$ if $x \in V(P_y)$; otherwise, let B denote the P_y-bridge of M^* containing x.

If $B = \{x\}$ let $x^* = x_q$. If $B \neq \{x\}$ let $x^* \in V(x_q R_q u'_q)$ with $x^* R_q x_q$ maximal such that x^* is contained in some $(x_q R_q u'_q \cup M^*)$-bridge of M with an attachment in $B - V(P_y \cap B)$. Let $u \in V(x^* R_q u'_q)$ with $x^* R_q u$ maximal such that u, y are incident with a common face of G. Clearly, $u = u'_q$ if $y = v''_q$.

Let K denote the union of x^*R_qu, M^*, and those $(M^* \cup x_qR_qu'_q)$-bridges of M whose attachments are all contained in $x^*R_qu \cup (\partial M^* - (V(B) - V(P_M)))$. By applying Lemma (2.5) (with $K, M^*, x^*R_qu, \partial M^*, P_y, y$ as K, L, Q, Q', T, u, respectively), we find path P'_x in $K - V(P_y)$ from x^* to u such that $P_y \cup P'_x$ is an x^*R_qu-Tutte subgraph of K and any P_y-bridge of M^* containing no edge of $x^*\partial M^*u$ is also a $(P_y \cup P'_x)$-bridge of K.

If $u = u'_q$ then let $P_x := P'_x$ and $P_M := P_x \cup P_y$; in this case, P_M is a ∂M-Tutte subgraph of M, except for the P_M-bridge containing $B \cup x_qR_qx^*$ which has three attachments (in $V(P_y \cap B) \cup \{x^*\}$). So assume $u \neq u'_q$. Then $x_q, y \in V(M')$ and $u \in V(x_qR_qu''_q)$. See Figure 14(b). Let M'' denote the $\{u, y\}$-bridge of M' containing $u''_q\partial G'v''_q$. Let $N := M'' + \{v, vy, vu\}$, where v is a new vertex, such that $u''_q\partial G'v''_q, v''_q, y, yv, v, vu, u''_q$ occur ∂N in clockwise order. By applying Lemma (3.3) (with y, v, u''_q, v''_q as u, v, v', u', respectively), we find a $u''_q\partial G'v''_q$-Tutte path P_N in N from y to $\{u''_q, v''_q\}$ containing $\{u''_q, v''_q, v\}$. Clearly, $yv, vu \in E(P_N)$. Let $P_u := P_N - \{y, v\}$. Then P_u is a path from u to $\{u''_q, v''_q\}$ containing $\{u''_q, v''_q\}$ and avoiding y.

Suppose P_u is from u to u''_q. Let $P'_u := \{u'_q\}$ if $u'_q = u''_q$; and otherwise, in the u''_q-bridge of M containing u'_q, we use Lemma (2.3) (to all its blocks) to find a $u''_qR_qu'_q$-Tutte path P'_u from u''_q to u'_q. Let $P_x := P'_x \cup P_u \cup P'_u$ and $P_M := P_x \cup P_y$. If P_u is from u to v''_q, the we find P'_v from v''_q to v'_q in the same way as for P'_u, and let $P_x := P'_x \cup P_u \cup P'_v$, and $P_M := P_x \cup P_y$.

Note that P_y is a 1-way infinite path from y_q, and P_x is a finite path from x^* to $\{u'_q, v'_q\}$. Clearly, P_M is a ∂M-Tutte subgraph, with the exception of the P_M-bridge of M containing $B \cup x_qR_qx^*$, which has three attachments (all in $V(P_y \cap B) \cup \{x^*\}$).

The rest of the proof of Theorem (1.2) is the same, no matter if P_M is from v'_q or u'_q (when q is odd) and no matter if P_x is from x^* to u'_q or v'_q. So we shall assume that

(6.2) if q is odd then P_M is from v'_q (Figure 14(a)), and if q is even then P_x is from x^* to v'_q (Figure 14(b)).

Recall the definitions of $(H_i, R_i, x_i, y_i, D_i)$ from (5.8) and (5.9). We now use $(H_i, R_i, x_i, y_i, D_i)$ to reduce the proof of Theorem (1.2) to a Tutte path problem.

(6.3) *Reduction through $\mathcal{F}$ when $\mathcal{F} \neq \emptyset$.*

First, we construct a sequence of (G_i, F'_i, e_i, f_i), $1 \leq i \leq n$.

If w is not defined let $G_1 = G$, $F'_1 := F_1$, e_1 be the edge of $v_1F'_1u_1$ incident with v_1, and f_1 be the edge of $e_1F'_1u_1$ incident with e_1. If w is defined, let u, v be two distinct neighbors of w on $v_1F_1u_1 - u_1$ such that v_1, v, u, u_1 occur on $v_1F_1u_1$ in order and, subject to this, v_1F_1v and v_1F_1u are minimal. Let $e_1 = uv$, $G_1 = G + e_1$, $F'_1 := uF_1v + e_1$, and let f_1 denote the edge of uF'_1u_1 incident with e_1. The existence of v, u, f_1 is guaranteed by the minimality of $|\beta \cap G|$ in (5.2)(ii) and of K' in (5.4). Let uv be added in the open disc bounded by $v_1\beta u_1 \cup v_1F_1u_1$.

Note that any vertex of $v_1F_1'e_1$ is connected to v_1 by a simple curve, say β', in the disc bounded $v_1\beta u_1 \cup v_1F_1'u_1$ such that $|\beta' \cap G| < |v_1\beta u_1 \cap G|$. Also note that $G_1 - F'' = H$, and $G_1 - \{e_1, f_1\}$ is a subgraph of $G - V(I_G(F_1) - V(F_1))$. Moreover, if G_1 has a 2-way infinite F_1'-Tutte path P_1 through e_1, then since G is 4-connected, either P_1 or $(P_1 - w_1w_2) + \{w, ww_1, ww_2\}$ is a spanning 2-way infinite path in G.

If $n = 1$, we stop. Now assume $n \geq 2$.

Suppose for some $1 \leq i \leq n-1$, we have constructed (G_j, F_j', e_j, f_j), $1 \leq j \leq i$, such that each G_j is a 2-connected 3-indivisible plane graph, $G - V(I_G(F_j) - V(F_j)) \subseteq G_j$ and $G_j \cap K'' = (G - V(I_G(F_j) - V(F_j))) \cap K''$, $G_j - \{e_j, f_j\}$ is a subgraph of G, e_j and f_j are edges of $v_jF_j'u_j$ incident with a common vertex p_j, $u_jF_jv_j \subseteq F_j' - p_j$, F_j' is contained in the disc bounded by F_j, every vertex of $v_jF_j'e_j \cap v_jF_j'f_j$ is connected to v_j by a simple curve β' in the closed disc by $v_j\beta u_j \cup v_jF_j'u_j$ such that $|\beta' \cap G| \leq |v_j\beta u_j \cap G|$. Moreover, for $2 \leq j \leq i$, if G_j has a 2-way F_j'-Tutte path through e_j then G_{j-1} has a 2-way infinite F_{j-1}'-Tutte path through e_{j-1}.

Let G_i^* denote the unique infinite block of $G_i - V(F_i')$, and let F_{i+1}^* denote the cycle bounding the face of G_i^* containing F_i. By the minimality of $|\beta \cap G|$ (in (5.2)(ii)) and K' (in (5.4)), there exist two disjoint paths from a component of $v_iF_i'u_i - e_i$ to G_i^* internally disjoint from $F_i' \cup G_i^*$.

Hence by Lemma (4.2), there exist $p_{i+1} \in V(v_iF_i'u_i - \{u_i, v_i\})$, distinct $x, y \in V(v_{i+1}F_{i+1}^*u_{i+1})$, and $e_{i+1} \in \{p_{i+1}x, p_{i+1}y\}$ such that v_{i+1}, x, y, u_{i+1} occur on $v_{i+1}F_{i+1}^*u_{i+1}$ in order and $N_{G_i}(xF_{i+1}^*y - \{x, y\}) \cap V(I_{G_i}(F_{i+1}^*)) \subseteq \{x, y, p_{i+1}\}$ and such that if $G_{i+1} := (G_i^* - V(I_{G_i}(F_{i+1}^*) - (V(F_{i+1}) \cup \{p_{i+1}\})) + \{p_{i+1}x, p_{i+1}y\}$ has a 2-way infinite $(yF_{i+1}^*x + \{p_{i+1}, p_{i+1}x, p_{i+1}y\})$-Tutte path through e_{i+1} then G_i contains a 2-way infinite F_i'-Tutte path through e_i. Let $F_{i+1}' := yF_{i+1}^*x + \{p_{i+1}, p_{i+1}x, p_{i+1}y\}$, and let f_{i+1} be the edge in $\{p_{i+1}x, p_{i+1}y\}$ other than e_{i+1}.

When two disjoint paths in the above application of Lemma (4.2) can be chosen from $v_iF_i'e_i$, we have $e_{i+1} = p_{i+1}x$ and $p_{i+1} \in V(v_iF_i'e_i)$. Therefore, by the choice of $e_1, \ldots, e_{i+1}$, each vertex of $v_{i+1}F_{i+1}'e_{i+1} \cap v_{i+1}F_{i+1}'f_{i+1}$ is connected to v_{i+1} by a simple curve β' inside the disc bounded by $v_{i+1}'\beta u_{i+1}' \cup v_{i+1}'F_{i+1}'u_{i+1}'$ such that $|\beta' \cap G| < |v_{i+1}'\beta u_{i+1}' \cap G|$. This completes the construction of (G_i, F_i', e_i, f_i), $1 \leq i \leq n$.

We now reduce (G_n, F_n', e_n, f_n) to $(H_1', R_1', D_1', e_1', f_1')$ as follows.

If $\{x_1, y_1\} \subseteq V(F_n)$ (Figure 11(a)), then let H_1' be obtained from G_n by deleting the finite $\{x_1, y_1\}$-bridge of G_n containing $x_1F_n'y_1$, but not deleting x_1 and y_1. Let R_1' be the 2-way infinite path in the boundary of the face of H_1' containing R, let $D_1' := y_1R_1'x_1$, and let $e_1' = e_n$ and $f_1' = f_n$. Note that $H_1' \cap K'' = H_1 \cap K''$.

Clearly, if H_1' has a 2-way infinite R_1'-Tutte path containing $\{x_1, y_1, e_1'\}$ then it is also a 2-way infinite F_n'-Tutte path in G_n containing e_n. Hence (from the above arguments) each G_i, $1 \leq i \leq n$, has a 2-way infinite F_i'-Tutte path through e_i. So G has a spanning 2-way infinite path.

Now assume $\{x_1, y_1\} \not\subseteq V(F_n)$. Let G_n^* denote the unique infinite block of $G_n - V(F_n')$, and let F_{n+1}^* denote the 2-way infinite path in the boundary of the face of G_n^* containing F_n'. See Figure 11(b).

By Lemma (4.3) there exist $p_{n+1} \in V(v_n F_n' u_n - \{u_n, v_n\})$, distinct $x, y \in V(v_{n+1} F_{n+1}^* u_{n+1})$, and $e_1' \in \{p_{n+1}x, p_{n+1}y\}$ such that v_{n+1}, x, y, u_{n+1} occur on $v_{n+1} F_{n+1}^* u_{n+1}$ in order, $N_{G_n}(xF_{n+1}^* y - \{x, y\}) - V(G_{n+1}^*) \subseteq \{p_{n+1}\}$, and if $H_1' := (G_{n+1}^* + \{p_{n+1}, p_{n+1}x, p_{n+1}y\}) + \{\text{edges of } G_n \text{ from } p_{n+1} \text{ to } xF_{n+1}^* y - \{x, y\}\}$ has a 2-way infinite $((F_{n+1}^* - V(xF_{n+1}^* y - \{x, y\})) + \{p_{n+1}, p_{n+1}x, p_{n+1}y\})$-Tutte path through e_1' then G_n contains a 2-way infinite F_n'-Tutte path through e_n. Let $R_1' := (F_{n+1}^* - V(xF_{n+1}^* y - \{x, y\})) + \{p_{n+1}, p_{n+1}x, p_{n+1}y\}$, $D_1' := y_1 R_1' x_1$, and let $f_1' \in \{p_{n+1}x, p_{n+1}y\} - \{e_1'\}$. Note that $v_1' = v_{n+1}$ and $u_1' = u_{n+1}$ (see Case 2 of (5.8)) and $H_1' \cap K'' = H_1 \cap K''$.

When two disjoint paths in the above application of Lemma (4.3) may be chosen to be from $v_{n+1} F_{n+1}^* x$, we can assume $e_1' = p_{n+1}x$. Therefore, by the choice of $e_1, \ldots, e_n, e_1'$, each vertex of $v_1' R_1' e_1' \cap v_1' R_1' f_1'$ is connected to v_{n+1} by a simple curve β' inside the disc bounded by $v_1' \beta u_1' \cup v_1' R_1' u_1'$ such that $|\beta' \cap G| \leq |v_1' \beta u_1' \cap G|$.

Clearly, if H_1' contains a 2-way infinite R_1'-Tutte path containing $\{x_1, y_1, e_1'\}$, then G_n contains a 2-way infinite F_n-Tutte path through e_n. Hence, by the above reduction, G has a spanning 2-way infinite path.

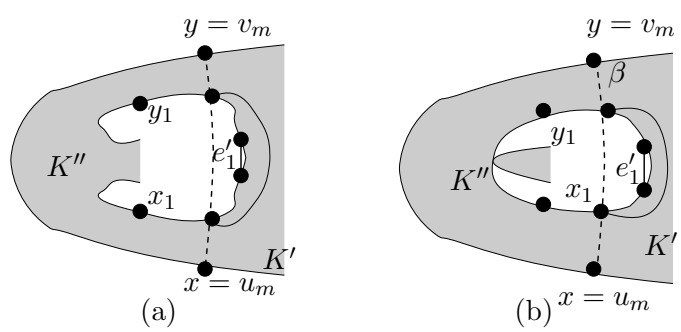


Figure 15: H_1', R_1', e_1'.

To summarize, we have constructed $(H_1', R_1', D_1', e_1', f_1')$ when $\mathcal{F} \neq \emptyset$, such that

(6.4) $H_1' \cap K'' = H_1 \cap K''$, and if H_1' contains a 2-way infinite R_1'-Tutte path containing $\{e_1', x_1, y_1\}$ then G has a spanning 2-way infinite path.

The reduction through $(H_i, R_i, x_i, y_i, D_i), 1 \leq i \leq q$ is based on the parity of q. In Section 7, we consider the case when q is odd, and the even case is treated in Section 8.

7 q is odd

In this section, we consider the case where q is odd. Recall P_M, B from (6.1). We refer to Figures 13, 14, and 15 for illustration.

Suppose $q \geq 3$ and $B \neq \{x_q\}$. We define $x'_1 = x_1$ when $x_1 \in S''$; define $x'_1 = w$ when $\mathcal{F} = \emptyset$, $w \in S''$, and $x_1 w \in E(G)$; and for all other situations, define $x'_1 = x_{q-1}$. Define $x^* \in V(x_{q-1}D_{q-1}u'_{q-1})$ so that x^* is contained in an $(H_q \cup D_{q-1})$-bridge of H_{q-1} and subject to this, $x_{q-1}D_{q-1}x^*$ is maximal. Let B^* denote the union of B, x'_1, $x_{q-1}R_{q-1}x^*$, all edges of G from x'_1 to $B \cup x_q R_{q-1} x_{q-1}$, and those $(H_q \cup D_{q-1})$-bridges of H_{q-1} whose attachments are all contained in $x_{q-1}R_{q-1}x^* \cup (B - V(P_M \cap B))$. Note that $B^* \subseteq K''$. Let $V(P_M \cap B) = \{b_1, b_2\}$ such that b_1, x_q, b_2 occur on R_q in order. By Lemma (3.4) (when $x'_1 \neq x_{q-1}$) and Lemma (2.4) (when $x'_1 = x_{q-1}$) and since G is 4-connected, we have

(7.1). If $q \geq 3$ and $B \neq \{x_q\}$, then (1) $B^* - \{b_1, b_2, x'_1\}$ has a spanning path P_B from x^* to $x_{q-1} \neq x'_1$, or (2) $B^* - \{b_1, b_2\}$ has a spanning path P_B from x^* to x'_1 through x_{q-1}.

Next, we extend P_M to a spanning 2-way infinite path in G. We consider two cases according (7.1).

(7.2) If $q = 1$, or $B = \{x_q\}$, or $q \geq 3$, $B \neq \{x_q\}$, and P_B in (7.1) is from x^* to x_{q-1}, then G has a spanning 2-way infinite path.

Proof. Our strategy is to construct a sequence of graphs H'_i, $1 \leq i \leq q$, find a 2-way infinite path in H'_q, and extend that path to paths in H'_i so that the path in H'_1 can be used to give a spanning 2-way infinite path in G. Recall the notation from (5.5), (5.8) and (5.9).

(a) *Construction of* $(H'_1, R'_1, D'_1, e'_1, f'_1)$.

When $\mathcal{F} \neq \emptyset$, $(H'_1, R'_1, D'_1, e'_1, f'_1)$ is defined as in (6.3). Now assume $\mathcal{F} = \emptyset$. In this case, $u'_i = u_i$ and $v'_i = v_i$ for $1 \leq i \leq q$.

Suppose w is not defined. Define $H'_1 := H_1 = H$ (see Case 2 in (5.8)), $R'_1 = R_1$, $D'_1 = D_1$, and let e'_1 denote the edge of $v'_1 D'_1 u'_1$ incident with v'_1.

Now suppose w is defined. Let v, u be distinct neighbors of w in $v'_1 R_1 u'_1 - u'_1$ such that v'_1, v, u, u'_1 occur on R_1 in order and, subject to this, $v'_1 R_1 v$ and $v'_1 R_1 u$ are minimal. Let $H'_1 := H_1 + uv$, $R'_1 := (R_1 - V(vR_1u - \{u, v\})) + uv$, $D'_1 := y_1 R'_1 x_1$, and $e'_1 := uv$. Let uv be added in the open disc bounded by $v'_1 \beta u'_1 \cup v'_1 R_1 u'_1$.

In both cases, we let f'_1 be the edge of $e'_1 D'_1 u'_1$ incident with e'_1. Note that each vertex of $v'_1 D'_1 e'_1 \cap v'_1 D'_1 f'_1$ is connected to v'_1 by a simple curve β' in the disc bounded by $v'_1 \beta u'_1 \cup v'_1 D'_1 u'_1$ such that $|\beta' \cap G| < |v'_1 \beta u'_1 \cap G|$.

(b) *Construction* of $(H'_i, R'_i, D'_i, e'_i, f'_i)$, $i = 1, \ldots, q$.

Suppose for some odd integer i, $1 \le i < q$, we have constructed $(H'_i, R'_i, D'_i, e'_i, f'_i)$, where H'_i is a 2-connected infinite plane graph, $H_i \subseteq H'_i$, $H'_i \cap K'' = H_i \cap K''$, R'_i is the 2-way infinite path in the boundary of the face of H'_i containing R, y_i, v'_i, u'_i, x_i occur on R'_i in order, $D'_i = y_i R'_i x_i$, $\{u'_i, v'_i\} \subseteq V(D'_i)$, $R_i - V(v'_i R_i u'_i - \{u'_i, v'_i\}) = R'_i - V(v'_i R'_i u'_i - \{u'_i, v'_i\}) \subseteq K''$, and e'_i and f'_i are consecutive edges of $v'_i D'_i u'_i$. Moreover, every vertex of $v'_i D'_i e'_i \cap v_i D'_i f'_i$ is connected to v'_i by a simple curve β' in the disc bounded by $v'_i \beta u'_i \cup v'_i D'_i u'_i$ such that $|\beta' \cap G| < |v'_i \beta u'_i \cap G|$.

(b1) *Construction of* $(H'_{i+1}, R'_{i+1}, D'_{i+1}, e'_{i+1}, f'_{i+1})$.

Let H''_i denote the unique infinite block of $H'_i - V(D'_i)$. Let R''_i be the 2-way infinite path in the boundary of the face of H''_i containing R. Note that $H''_i \cap K'' = H_{i+1} \cap K''$, $y_{i+1}, v'_{i+1}, u'_{i+1}, x_{i+1}$ occur on R''_i in order, and $R''_i - V(v'_{i+1} R''_i u'_{i+1} - \{u'_{i+1}, v'_{i+1}\}) = R_{i+1} - V(v'_{i+1} R_{i+1} u'_{i+1} - \{u'_{i+1}, v'_{i+1}\})$. Let $w_1, \ldots, w_p$ be the attachments on R''_i of $(H''_i \cup D'_i)$-bridges of H'_i, and let $s_j, t_j \in V(D'_i)$ with $s_j D'_i t_j$ maximal, such that $\{s_j, w_j\}$ and $\{t_j, w_j\}$ are each contained in an $(H''_i \cup D'_i)$-bridge of H'_i. Without loss of generality we may assume that $s_1, t_1, \ldots, s_p, t_p$ occur in this order on D'_i, with $s_1 = y_i$ and $t_p = x_i$. Clearly, $w_1 = y_{i+1}$ and $w_p = x_{i+1}$.

By the choice of β in (5.2)(ii), there exists $w_l \in V(v'_{i+1} R''_i u'_{i+1})$ such that either $e'_i \in E(s_l D'_i s_{l+1})$ or $e'_i \in E(t_l D'_i t_{l-1})$. We choose such w_l so that $v'_{i+1} R''_i w_l$ is minimal. Let e'_{i+1} be the edge of $v'_{i+1} R''_i u'_{i+1}$ incident with w_l such that $v_{i+1} R''_i e'_{i+1}$ is minimal, and let f'_{i+1} be the edge in $e'_{i+1} R''_i u'_{i+1}$ incident with e'_{i+1}. Let $H'_{i+1} := H''_i$ and $R'_{i+1} := R''_i$, and let $D'_{i+1} := y_{i+1} R'_{i+1} x_{i+1}$. Clearly, $H_{i+1} \subseteq H'_{i+1}$ and $H'_{i+1} \cap K'' = H_{i+1} \cap K''$. This completes the construction of $(H'_{i+1}, R'_{i+1}, D'_{i+1}, e'_{i+1}, f'_{i+1})$.

From the choice of w_l and the properties of vertices of $v'_i D'_i e'_i \cap v'_i D'_i f'_i$, we see that every vertex of $v'_{i+1} D'_{i+1} e'_{i+1} \cap v'_{i+1} D'_{i+1} f'_{i+1}$ is connected to v'_{i+1} by a simple curve β' in the disc bounded by $v'_{i+1} \beta u'_{i+1} \cup v'_{i+1} D'_{i+1} u'_{i+1}$ such that $|\beta' \cap G| < |v'_{i+1} \beta u'_{i+1} \cap G|$.

Claim. If H'_{i+1} contains disjoint 1-way infinite paths X_{i+1}, Y_{i+1} from x_{i+1}, y_{i+1}, respectively, such that $X_{i+1} \cup Y_{i+1}$ is an R'_{i+1}-Tutte subgraph of H'_{i+1} through e'_{i+1}, then H'_i has a 2-way infinite R'_i-Tutte path P'_i containing $X_{i+1} \cup Y_{i+1}$ and $\{e'_i, x_i, y_i\}$.

To prove this claim, we need a path from x_{i+1} to s_p through x_i, a path from y_{i+1} to t_1 through y_i, and a path from t_1 to s_p.

Let J_s denote the union of $s_p D'_i t_p$ and those $(H'_{i+1} \cup D'_i)$-bridges of H'_i whose attachments are all contained in $s_p D'_i t_p + x_{i+1}$. In $J_s + s_p x_{i+1}$, we apply Lemma (2.4) to find an $x_{i+1} R'_i s_p$-Tutte path P_s from x_{i+1} to s_p through x_i and e'_i (if $e'_i \in E(J_s)$).

Let J_t denote the union of $s_1 D'_i t_1$ and those $(H'_{i+1} \cup D'_i)$-bridges of H'_i whose attachments are all contained in $s_1 D'_i t_1 + y_{i+1}$. In $J_t + t_1 y_{i+1}$, we apply Lemma (2.4) to find a $y_{i+1} R'_i t_1$-Tutte path P_t from y_{i+1} to t_1 through y_i and e'_i (if $e'_i \in E(J_t)$).

Let $K := H'_i - V((J_s - \{x_{i+1}, s_p\}) \cup (J_t - \{y_{i+1}, t_1\}))$. By applying Lemma (2.5) to K (with $K, H'_{i+1}, t_1 D'_i s_p, R'_{i+1}, X_{i+1} \cup Y_{i+1}, w$ as K, L, Q, Q', T, u, respectively), we obtain a path S from t_1 to s_p in $K - V(X_{i+1} \cup Y_{i+1})$ such that $X_{i+1} \cup Y_{i+1} \cup S$ is a $t_1 D'_i s_p$-Tutte

subgraph of K, and every $(X_{i+1} \cup Y_{i+1})$-bridge of H'_{i+1} containing no edge of R'_{i+1} is also an $(X_{i+1} \cup Y_{i+1} \cup S)$-bridge of K.

Next, we modify S. Let $S' := S$ if $e'_i \notin E(t_1 D'_i s_p)$. Now suppose $e'_i \in E(t_1 D'_i s_p)$. We consider the case $e'_i \in E(s_l D'_i s_{l+1})$; the case $e'_i \in E(t_l D'_i t_{l-1})$ is similar. Let W denote the union of $s_l D'_i s_{l+1}$ and those $(H'_{i+1} \cup D'_i)$-bridges of H'_i whose attachments are all contained in $s_l D'_i s_{l+1} + w_l$. Note that $s_l S s_{l+1} \subseteq W - w_l$. In $W + w_l s_{l+1}$ we apply Lemma (2.4) to find an $s_l D'_i s_{l+1}$-Tutte path S'' from s_l to w_l through e'_i and $w_l s_{l+1}$, and let S' be obtained from S by replacing $s_l S s_{l+1}$ with $S'' - w_l$.

It is now easy to verify that $P'_i := X_{i+1} \cup P_s \cup S' \cup P_t \cup Y_{i+1}$ is the desired 2-way infinite R'_i-Tutte path in H'_i. This proves the above claim.

(b2) *Construction of* $(H'_{i+2}, R'_{i+2}, D'_{i+2}, e'_{i+2}, f'_{i+2})$.

Let H''_{i+1} be the minimal union of blocks of $H'_{i+1} - V(D'_{i+1})$ containing all infinite blocks of $H'_{i+1} - V(D'_{i+1})$. Since $H_{i+1} \subseteq H'_{i+1}$, we have $H_{i+2} \subseteq H''_{i+1}$. Let R''_{i+1} be the maximal subgraph of H''_{i+1} in the boundary of the face of H''_{i+1} containing R. Since $H'_{i+1} \cap K'' = H_{i+1} \cap K''$, we have $H''_{i+1} \cap K'' = H_{i+2} \cap K''$. Note that $y_{i+2}, v'_{i+2}, u'_{i+2}, x_{i+2} \in V(R''_{i+1})$. Moreover, if H''_{i+1} has two infinite blocks, then R''_{i+1} contains a facial cycle D''_{i+1} of the infinite block of H''_{i+1} intersecting K', $\{v'_{i+2}, u'_{i+2}\} \subseteq V(D''_{i+1})$, and $v'_{i+2} D''_{i+1} u'_{i+2} \subseteq K'$. For convenience, for any $a, b \in V(v'_{i+2} D''_{i+1} u'_{i+2})$ with v'_{i+2}, a, b, u'_{i+2} on D''_{i+1} in cyclic order, we write $v'_{i+2} R''_{i+1} u'_{i+2} = v'_{i+2} D''_{i+1} u'_{i+2}$.

We wish to apply Lemma (4.3). Let H' be the 2-connected infinite plane graph obtained from H'_{i+1} by adding the edge $x_{i+1} y_{i+1}$ (in the face of H'_{i+1} containing D'_i), such that $F' := D'_{i+1} + x_{i+1} y_{i+1}$ is a facial cycle of H'. Because of the choice of β in (5.2)(ii), there exist two disjoint paths in H' from a component of $v'_{i+1} F' u'_{i+1} - e'_{i+1}$ to $v'_{i+2} R''_{i+1} u'_{i+2}$ and internally disjoint from $F' \cup H''_{i+1}$.

Then by Lemma (4.3) there exist $p \in V(v'_{i+1} F' u'_{i+1} - \{u'_{i+1}, v'_{i+1}\})$, distinct $x, y \in V(v'_{i+2} R''_{i+1} u'_{i+2})$, and $e'_{i+2} \in \{px, py\}$ such that v'_{i+2}, x, y, u'_{i+2} occur on $v'_{i+2} R''_{i+1} u'_{i+2}$ in order and $N_{H'}(x R''_{i+1} y - \{x, y\}) - V(H''_{i+1}) \subseteq \{p\}$, and if $H'_{i+2} := (H''_{i+1} + \{p, px, py\}) + \{\text{edges of } H' \text{ from } p \text{ to } x R''_{i+1} y - \{x, y\}\}$ has a 2-way infinite $(R''_{i+1} - V(x R''_{i+1} y - \{x, y\})) + \{p, px, py\})$-Tutte path P'_{i+2} through e'_{i+2} then H' contains a 2-way infinite F'-Tutte path P' through e'_{i+1} such that $P'_{i+2} - p \subseteq P'$. Let $f'_{i+2} \in \{px, py\} - \{e'_{i+2}\}$. Let px, py be added in the open disc bounded by $v'_{i+2} \beta u'_{i+2} \cup v'_{i+2} R''_{i+1} u'_{i+2}$ so that R is contained in a face of H'_{i+2}.

Note that, if in the above application of Lemma (4.3) the two disjoint paths may be chosen to be from $v'_{i+1} F' e'_{i+1}$ then we have $e'_{i+2} = px$. Therefore, every vertex of $v'_{i+2} R''_{i+1} e'_{i+2} \cap v'_{i+2} R''_{i+1} f'_{i+2}$ is connected to v'_{i+2} by a simple curve β' in the disc bounded by $v'_{i+2} \beta u'_{i+2} \cup v'_{i+2} R''_{i+1} u'_{i+2}$ such that $|\beta' \cap G| < |v'_{i+2} \beta u'_{i+2} \cap G|$.

Let $R'_{i+2} := (R''_{i+1} - V(x R''_{i+1} y - \{x, y\})) + \{p, px, py\})$, which lies in the boundary of the face of H'_{i+2} containing R. Let $D'_{i+2} := y_{i+2} R'_{i+2} x_{i+2}$ when H'_{i+2} is 2-connected; otherwise, let $D'_{i+2} = \emptyset$.

Claim. If H'_{i+2} contains a 2-way infinite R'_{i+2}-Tutte path P'_{i+2} through y_{i+2} and e'_{i+2} and if $P_M \subseteq P'_{i+2}$ when $i+2=q$, then there are disjoint 1-way infinite paths X_{i+1}, Y_{i+1} in H'_{i+1} from x_{i+1}, y_{i+1}, respectively, such that $X_{i+1} \cup Y_{i+1}$ is an R'_{i+1}-Tutte subgraph of H'_{i+1} containing e'_{i+1}.

To see this, we note from the above argument that under the hypothesis of the above claim, H' contains a 2-way infinite F'-Tutte path P' through e'_{i+1} such that $P'_{i+2}-p \subseteq P'$. Therefore, since $y_{i+2} \in V(P'_{i+2})$, $x_{i+1}y_{i+1} \in E(P')$, where $x_{i+1}y_{i+1}$ is added in order to apply Lemma (4.3). Let X_{i+1}, Y_{i+1} be the component of $P' - x_{i+1}y_{i+1}$ containing x_{i+1}, y_{i+1}, respectively. When $i+2<q$, X_{i+1}, Y_{i+1} give the desired paths for the above claim. So assume $i+2=q$. Then $x^*X_{q-1}x_{q-1} \subseteq B^* - \{x'_1, b_1, b_2\}$ since $P_M \subseteq P'$. So we can modify X_{i+1} such that $x^*X_{q-1}x_{q-1} = P_B$. Then X_{q-1} and Y_{q-1} give the desired 1-way infinite paths in H'_{q-1} for the above claim.

(c) H'_q contains a 2-way infinite R'_q-Tutte path P'_q through e'_q such that $P_M \subseteq P'_q$. (Since $P_M \subseteq P'_q$, if $B \neq \{x_q\}$ then B is also the P'_q-bridge of H'_q containing x_q.)

Suppose H'_q has two infinite blocks. See Figure 13(a). Let L denote the infinite block of H'_q that intersects K'. In L, we use Lemma (2.7) to find a 1-way infinite $(R'_q \cap L)$-Tutte path P_L from v' through e'_q. Then $P_M \cup P_L$ gives the desired P'_q for (c).

Now assume H'_q has a unique infinite block. See Figure 14(a). Then $\{u'_q, v'_q\} = \{u_m, v_m\}$ and P_M is a 1-way infinite path from v'_q containing $\{u''_q, y_q\}$. Let L denote the $\{u''_q, v'_q\}$-bridge of H'_q intersecting K'. Let $e'_q = u'v'$ such that v'_q, v', u', u'_q occur on R'_q in order. Recall that every vertex of $v'_qR'_qe'_q \cap v'_qR'_qf'_q$ is connected to v'_q by a simple curve β' in the closed disc bounded by $v'_q\beta u'_q \cup v'_qR'_qu'_q$ such that $|\beta' \cap G| < |v'_q\beta u'_q \cap G|$. Also, v'_q, e'_q, f'_q, u'_q occur on $v'_qR'_qu'_q$ in this order, unless there do not exist two disjoint paths from $v'_{q-1}R'_{q-1}u'_{q-1}$ to H''_{q-1} internally disjoint from $D'_{q-1} \cup H''_{q-1}$. Thus, by the minimality of $|\beta \cap G|$ in (5.2)(ii) and of K' in (5.4), $L - u''_q$ has a 1-way infinite path from v'_q through e'_q. We apply Lemma (2.7) to $L + u''_qv'_q$ and find a 1-way infinite $(v'_qR'_qu''_q + v'_qu''_q)$-Tutte path P_L from u''_q through e'_q such that $v' \in V(u''_qP_Lu')$. By planarity, $u''_qv'_q \in E(P_L)$. Now $P'_q := P_M \cup (P_L - u''_q)$ is the desired path for (c).

Since $y_q \in V(P_M)$, $y_q \in V(P'_q)$. By applying claims in (b1) and (b2), we conclude that H'_1 contains a 2-way infinite Tutte path P'_1 containing $\{e'_1, x_1, y_1\}$. If $\mathcal{F} \neq \emptyset$, then by (6.4) G has a spanning 2-way infinite path. Now assume $\mathcal{F} = \emptyset$. Then by (a) and since G is 4-connected, P'_1 or $(P'_1 - w) + \{w, wu, wv\}$ is a spanning 2-way infinite path in G. □

(7.3) If $q \geq 3$, $B \neq \{x_q\}$, $x'_1 \neq x_{q-1}$, and P_B in (7.1) is from x^* to x'_1 through x_{q-1}, then G has a spanning 2-way infinite path.

Proof. We use a similar approach as in (7.2) to complete $P_M \cup P_B$ to the desired path.

(a) *Construction of* $(H_1, R'_1, D'_1, e'_1, f'_1)$.

When $\mathcal{F} \neq \emptyset$, $(H_1', R_1', D_1', e_1', f_1')$ is defined as in (6.3). Now assume $\mathcal{F} = \emptyset$.

Suppose w is not defined or $x_1' = w$. Define $H_1' := H_1 = H$ (see Case 2 in (5.8)), $R_1' := R_1$, $D_1' := D_1$, and let e_1' denote the edge of $v_1'D_1'u_1'$ incident with v_1'.

Now suppose w is defined and $x_1' \neq w$. Let v, u be distinct neighbors of w in $v_1'R_1u_1' - u_1'$ such that v_1', v, u, u_1' occur on R_1 in order and, subject to this, $v_1'R_1v$ and $v_1'R_1u$ are minimal. Let $H_1' := H_1 + uv$, $R_1' := (R_1 - V(vR_1u - \{u, v\})) + uv$, $D_1' := y_1R_1'x_1$, and $e_1' := uv$. Let uv be added in the open disc bounded by $v_1'\beta u_1' \cup v_1'R_1u_1'$.

In all above cases, we let f_1' be the edge of $e_1'D_1'u_1'$ incident with e_1'. Note that each vertex of $v_1'R_1'e_1' \cap v_1'D_1'f_1'$ is connected to v_1' by a simple curve β' in the disc bounded by $\beta \cup v_1'D_1'u_1'$ such that $|\beta' \cap G| < |v_1'\beta u_1' \cap G|$.

(b) *Construction* of $(H_i', R_i', D_i', e_i', f_i')$, $1 \leq i \leq q$.

Suppose we have constructed $(H_i', R_i', D_i', e_i', f_i')$ for some integer i, where $1 \leq i < q$, H_i' is a 2-connected 3-indivisible infinite plane graph, $H_i \subseteq H_i'$ and $H_i \cap K'' = H_i' \cap K''$, R_i' is the 2-way infinite path in the boundary of the face of H_i' containing R, y_i, v_i', u_i', x_i occur on R_i' in this order, $R_i - V(v_i'R_iu_i' - \{u_i', v_i'\}) = R_i' - V(v_i'R_i'u_i' - \{u_i', v_i'\}) \subseteq K''$, $D_i' = y_iR_i'x_i$, and e_i', f_i' are consecutive edges on $v_i'D_i'u_i'$. Moreover, each vertex of $v_i'D_i'e_i' \cap v_i'D_i'f_i'$ is connected to v_i' by a simple curve β' in the disc bounded by $v_i'\beta u_i' \cup v_i'D_i'u_i'$ such that $|\beta' \cap G| < |v_i'\beta u_i' \cap G|$.

(b1) For $1 \leq i \leq q-2$, we use exactly the same construction in (b1) of (7.2) to construct $(H_{i+1}', R_{i+1}', D_{i+1}', e_{i+1}', f_{i+1}')$ from $(H_i', R_i', D_i', e_i', f_i')$.

Let $H_i^* := H_i' \cup B^*$ for $1 \leq i \leq q-1$, let R' be the face of H_i^* intersecting R but not bounded by a cycle, and let R_i^* denote the maximal subgraph of H_i^* in the boundary of R'.

Claim. If i is odd (respectively, even) and H_{i+1}^* contains disjoint 1-way infinite paths X_1' from x_1' and Z_{i+1} from y_{i+1} (respectively, x_{i+1}), such that $X_1' \cup Z_{i+1}$ is an R_{i+1}^*-Tutte subgraph of H_{i+1}^* containing $\{e_{i+1}', x_{i+1}, y_{i+1}\}$, then H_i^* contains 1-way infinite R_i^*-Tutte path Z_i from x_i (respectively, y_i) containing Z_{i+1} and $\{e_i', x_i, y_i\}$ such that $X_1' \cap Z_i \subseteq \{x_1'\} \cap \{x_1\}$ and $X_1' \cup Z_i$ is an R_i^*-Tutte subgraph of H_i^*.

We deal with the case when i is odd; the other case can be dealt with in exactly the same way. Let H_i'' denote the unique infinite block of $H_i' - V(D_i')$.

Let $t \in V(D_i')$ with $tD_i'y_i$ maximal such that $\{t, y_{i+1}\}$ is contained in some $(H_i'' \cup D_i')$-bridge of H_i'. Let J_t denote the union of $tD_i'y_i$ and those $(H_i'' \cup D_i')$-bridges of H_i' whose attachments are all contained in $tD_i'y_i + y_{i+1}$. In $J_t + y_{i+1}t$ we use Lemma (2.4) to find a $y_{i+1}R_i't$-Tutte path S_t from y_{i+1} to t through y_i and e_i' (if $e_i' \in E(tD_i'y_i)$).

Let $s \in V(D_i')$ with $sD_i'x_i$ maximal such that $\{s, x_{i+1}\}$ is contained in some $(H_i'' \cup D_i')$-bridge of H_i'. Let J_s denote the union of $sD_i'x_i$ and those $(H_i'' \cup D_i')$-bridges of H_i' whose attachments are all contained in $sD_i'x_i + x_{i+1}$. In $J_s + x_{i+1}s$, we use Lemma (2.3) to find an $x_{i+1}R_i's$-Tutte path S_s' from s to x_i through $x_{i+1}s$ and e_i' (if $e_i' \in E(sD_i'x_i)$); and

let $S_s = S'_s - x_{i+1}$. From the choice of w_l (in the paragraph before the claim in (b1) of (7.2)) and by the minimality of K' in (5.4), we have $e'_i \notin E(J_s)$.

Let $H^* := H^*_i - V((J_t - \{y_{i+1}, t\}) \cup (J_s - \{x_{i+1}, s\}))$. We apply Lemma (2.5), with $H^*, H^*_{i+1}, tD'_is, R^*_{i+1}, X'_1 \cup Z_{i+1}, x_{i+1}$ as K, L, Q, Q', T, u, respectively. There is a path S in $H^* - V(X'_1 \cup Z_{i+1})$ from t to s such that $S \cup X'_1 \cup Z_{i+1}$ is a tD'_is-Tutte subgraph of H^*, and every $(X'_1 \cup Z_{i+1})$-bridge of H^*_{i+1} containing no edge of D'_{i+1} is an $(S \cup X'_1 \cup Z_{i+1})$-bridge of H^*. Moreover, because $e'_{i+1} \in E(X'_1 \cup Z_{i+1})$, we may assume, as in (b1) of (7.2), $e'_i \in E(S)$ (if $e'_i \in E(tD'_is)$).

Let $Z_i := Z_{i+1} \cup S_t \cup S \cup S_s$. Then it is easy to verify that Z_i is the desired path for the above claim.

(b2) Now we consider the case where $1 \leq i = q - 1$.

Let H''_{q-1} be the minimal union of blocks in $H'_{q-1} - V(D'_{q-1})$ containing all infinite blocks of $H'_{q-1} - V(D'_{q-1})$, and R''_{q-1} be the maximal subgraph of H''_{q-1} in the boundary of the face of H''_{q-1} containing R.

We make use of Lemma (4.3). Define $H^* := (H^*_{q-1} - V(B^* - V(B + x^*))) + x^*y_{q-1}$. Let $F^* := x^*D'_{q-1}y_{q-1} + x^*y_{q-1}$. By the choices of β in (5.2)(ii) and of K' in (5.4), there exist two disjoint paths in H^* from a component of $v'_{q-1}D'_{q-1}u'_{q-1} - e'_{q-1}$ to $v'_qR''_{q-1}u'_q$ internally disjoint from $F^* \cup H''_{q-1}$.

Then by Lemma (4.3) there exist $p \in V(v'_{q-1}F^*u'_{q-1} - \{u'_{q-1}, v'_{q-1}\})$, distinct $x, y \in V(v'_qR''_{q-1}u'_q)$, and $e'_q \in \{px, py\}$ such that v'_q, x, y, u'_q occur on $v'_qR''_{q-1}u'_q$ in order and $N_{H^*}(xR''_{q-1}y - \{x, y\}) - V(H''_{q-1}) \subseteq \{p\}$, and if $H'_q := H''_{q-1} + \{p, px, py\} + \{$all edges of G from p to $xR''_{q-1}y - \{x, y\}\}$ has a 2-way infinite $((R''_{q-1} - V(xR''_{q-1}y - \{x, y\})) + \{p, px, py\})$-Tutte path P'_q through e'_q then H^* contains a 2-way infinite F^*_1-Tutte path P^* through e'_{q-1} such that $P'_q - p \subseteq P^*$. Let $f'_q \in \{px, py\} - \{e'_q\}$.

Let $R'_q := (R''_{q-1} - V(xR''_{q-1}y - \{x, y\})) + \{p, px, py\}$ denote the maximal subgraph of H'_q in the boundary of the face of H'_q containing R, and let $D'_q := (y_qR''_{q-1}x_q - V(xR''_{q-1}y - \{x, y\})) + \{p, px, py\}$. Note that we may choose $e'_q = px$ unless there do not exist two disjoint paths from $v'_{q-1}D'_{q-1}e'_{q-1}$ to H''_q internally disjoint from $F^* \cup H''_q$. Hence, every vertex of $v'_qR'_qe'_q \cap v'_qR'_qf'_q$ is connected to v'_q by a simple curve β' in the disc bounded by $v'_q\beta u'_q \cup v'_qR'_qu'_q$ such that $|\beta' \cap G| < |v'_q\beta u'_q \cap G|$.

By exactly the same argument used for (c) of (7.2), H'_q contains a 2-way infinite R'_q-Tutte path P'_q through e'_q, such that $P_M \subseteq P'_q$. Hence, by our assumption of P_M, $y_q \in V(P'_q)$ and B is the P'_q-bridge of H'_q containing x_q. Thus by above conclusion, H^* contains a 2-way infinite F^*-Tutte path P^* through e'_{q-1} such that $P'_q - p \subseteq P^*$. Clearly, P^* uses x^*y_{q-1}. Let X^* and Z_{q-1} denote the 1-way infinite paths in $P^* - x^*y_{q-1}$ containing x^*, y_{q-1}, respectively. Now $X'_1 := X^* \cup P_B$ and Z_{q-1} are the desired paths for H^*_{q-1}.

Therefore, using P'_q and by the claim in (b1), H^*_1 contains 1-way infinite paths X'_1, Z_1 such that $X'_1 \cap Z_1 \subseteq \{x'_1\} \cap \{x_1\}$ and $X'_1 \cup Z_1$ is an R^*_1-Tutte subgraph of H^*_1. If $\mathcal{F} \neq \emptyset$

then $x_1' = x_1$ and $X_1' \cup Z_1$ is a 2-way infinite R_1'-Tutte path in H_1', and by (6.4), G has a spanning 2-way infinite path. Now assume $\mathcal{F} = \emptyset$. If $x_1' = x_1$ then again $X_1' \cup Z_1$ is a 2-way infinite R_1'-Tutte path in H_1'; and since G is 4-connected, it is a spanning 2-way infinite path in G. So assume $x_1' = w \neq x_1$. Then $(X_1' \cup Z_1) + wx_1$ is a 2-way infinite R_1^*-Tutte path in H_1^*. Since $V(H_1^*) = V(G)$ and G is 4-connected, it is a spanning 2-way infinite path in G. □

8 q is even

In this section, we complete the proof of Theorem (1.2) by considering the case where q is even (and hence $q \geq 2$). The argument is similar to that in Section 7. Recall P_M, B, x, x^* from (6.1), and see Figures 13(a), 14(a) and 15 .

Let $x_1' = x_1$ if $x_1 \in S''$; let $x_1' = w$ if $\mathcal{F} = \emptyset$, $w \in S''$, and $x_1 w \in E(G)$; for other situations, let $x_1' = x_q$. Let B^* be obtained from the $\{b_1, b_2, x^*\}$-bridge of H_q containing $B + x_q R_q x^*$ by adding x_1' and all edges of G from x_1' to $B \cup x R_q x^*$. When $B \neq \{x\}$, let $\{b_1, b_2\} = V(P_M \cap B)$. By Lemma (3.4) and since G is 4-connected, we have

(8.1). If $B \neq \{x\}$, then (1) $B^* - \{b_1, b_2, x_1'\}$ has a spanning path P_B from x^* to x_q, or (2) $B^* - \{b_1, b_2\}$ has a spanning path P_B from x^* to x_1' through x_q.

As in section 7, we distinguish two cases according to (8.1).

(8.2) *If $B = \{x\}$ or P_B in (8.1) is a path from x^* to x_q, then G has a spanning 2-way infinite path.*

Proof. The argument here is essentially the same as for (7.2); so we only give a sketch. We first construct $(H_1', R_1', D_1', e_1', f_1')$ as in (a) of (7.2). Now suppose we have constructed $(H_i', R_i', D_i', e_i', f_i')$ for some odd integer i, $1 \leq i < q$. As in (b1) of (7.2) we construct $(H_{i+1}', R_{i+1}', D_{i+1}', e_{i+1}', f_{i+1}')$ and we have

Claim. If H_{i+1}' contains disjoint 1-way infinite paths X_{i+1}, Y_{i+1} from x_{i+1}, y_{i+1}, respectively, such that $X_{i+1} \cup Y_{i+1}$ is an R_{i+1}'-Tutte subgraph of H_{i+1}' containing e_{i+1}', then H_i' contains a 2-way infinite R_i'-Tutte path P_i' containing $X_{i+1} \cup Y_{i+1}$ and e_{i+1}'.

As in (b2) of (7.2), we construct $(H_{i+2}', R_{i+2}', D_{i+2}', e_{i+2}', f_{i+2}')$ and we have

Claim. If H_{i+2}' has a 2-way infinite R_{i+2}'-Tutte path P_{i+2}' containing $\{x_{i+2}, y_{i+2}, e_{i+2}'\}$, then there are disjoint 1-way infinite paths X_{i+1}, Y_{i+1} in H_{i+1}' from x_{i+1}, y_{i+1}, respectively, such that $X_{i+1} \cup Y_{i+1}$ is an R_{i+1}'-Tutte subgraph of H_{i+1}' containing e_{i+1}'.

Next we show that H_q' contains disjoint 1-way infinite paths X_q and Y_q from x_q and y_q, respectively, such that $X_q \cup Y_q$ is an R_q'-Tutte subgraph of H_q' containing e_q'.

If H_q' has two infinite blocks, then let $L = M'$. See Figure 13(a)(c). In L, we use Lemma (2.7) to find a 1-way infinite $(R_q' \cap L)$-Tutte path P_L from v' through e_q'. Clearly, $P_L \cup P_M \cup P_B$ is the union of the desired disjoint 1-way infinite paths X_q, Y_q from x_q, y_q, respectively.

Now assume H'_q is 2-connected. See Figure 14(b). Let L denote the $\{u''_q, v'_q\}$-bridge of H'_q intersecting K'. We use exactly the same argument as in (c) of (7.2) to find a 1-way infinite $(R'_q \cap L)$-Tutte path P_L in $L + u''_q v'_q$ from u''_q through e'_q and $u''_q v'_q$. Then $(P_L - u''_q) \cup P_M \cup P_B$ is the unioon of the desired disjoint 1-way finite paths X_q, Y_q from x_q, y_q, respectively.

Finally, by the same argument as in the end of (7.2), we conclude that G has a spanning 2-way infinite path. □

(8.3) *If $B \neq \{x\}$, $x'_1 \neq x_q$, and P_B in (8.1) is a path from x^* to x'_1 through x_q, then G has a spanning 2-way infinite path.*

Proof. We proceed as in (7.3). First, we construct $(H'_1, R'_1, D'_1, e'_1, f'_1)$ as in (a) of (7.3). Then, as in (b1) of (7.3), we construct $(H'_i, R'_i, D'_i, e'_i, f'_i)$ for $1 \leq i \leq q-1$. Let $H^*_i := H'_i \cup B^*$, and let R^*_i denote the subgraph of H^*_i in the boundary of the face of H^*_i intersecting R but not bounded by a cycle. Again as in (b1) of (7.3), we have

Claim. For $1 \leq i \leq q-2$, if i is odd (respectively, even) and H^*_{i+1} contains disjoint 1-way infinite paths X'_1 and Z_{i+1} from x'_1 and y_{i+1} (respectively, x_{i+1}), respectively, such that $X'_1 \cup Z_{i+1}$ is an R^*_{i+1}-Tutte subgraph of H^*_{i+1} containing $\{e'_{i+1}, x_{i+1}, y_{i+1}\}$, then H^*_i contains a 1-way infinite R^*_i-Tutte path Z_i from x_i (respectively, y_i) containing Z_{i+1} and e'_i such that $X'_1 \cap Z_i \subseteq \{x'_1\} \cap \{x_1\}$ and $X'_1 \cup Z_i$ is an R^*_i-Tutte subgraph of H^*_i.

By the argument in (b2) and (c) of (7.3), we can show that H^*_q (respectively, H^*_{q-1}) contains a 1-way infinite path Z_q (respectively, Z_{q-1}) from y_q (respectively, x_{q-1}) such that $X'_1 \cap Z_q \subseteq \{x'_1\} \cap \{x_1\}$ and $X'_1 \cup Z_q$ is an R^*_q-Tutte subgraph of H^*_q containing e'_q, and $X'_1 \cap Z_{q-1} = \emptyset$ and $X'_1 \cup Z_{q-1}$ is an R^*_{q-1}-Tutte subgraph of H^*_{q-1}.

Therefore, the existence of a spanning 2-way infinite path in G follows from the above claim and the arguments in the end of the proof of (7.3). □

We now have completed the proof of Theorem (1.2).

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