Comment on "A proof for a conjecture on the Randić index of graphs with diameter"*

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Abstract

Aouchiche et al. proposed a conjecture on the relationship between the Randić index $R(G)$ and the diameter $D(G)$ of a graph G: $R(G) - D(G) \geq \sqrt{2} - \frac{n+1}{2}$ and $\frac{R(G)}{D(G)} \geq \frac{n-3+2\sqrt{2}}{2n-2}$, with equalities if and only if G is a path. In [Jianxi Liu, Meili Liang, Bo Cheng, Bolian Liu, A proof for a conjecture on the Randić index of graphs with diameter. *Appl. Math. Lett.* $\mathbf{24}(2011), 752-756$], the authors claimed that they proved the first part of the conjecture. But we find that their proof is invalid. This paper is to point out that the proofs of their two crucial lemmas are incorrect. Moreover, we give two counterexamples to show that the conclusions of their two lemmas are essentially wrong, which means they did not provide a proof for the first part of the conjecture.

Keywords: Randić index, diameter, conjecture

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1 Introduction

The Randić index $R(G)$ of a graph G is defined as follows:

$$R = R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}, \tag{1}$$

where $d(u)$ denotes the degree of a vertex u and the summation runs over all edges uv of G. The distance between two vertices u and v in G, denoted by $d_G(u,v)$ (or $d(u,v)$ for short), is the length of a shortest path connecting u and v in G. The diameter $D(G)$ of G is the maximum distance $d(u,v)$

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over all pairs of vertices u and v of G. A pendent vertex (or leaf) is a vertex of degree 1. An edge incident with a leaf is called a pendent edge. If $E' \subseteq E(G)$, then $G - E' = (V(G), E(G) - E')$. If $|G| > 1$ and $G - E'$ is connected for every set $E' \subseteq E$ of fewer than l edges, then G is called l-edge-connected. The greatest integer l such that G is l-edge-connected is the edge-connectivity $\lambda(G)$ of G. The weight of an edge $uv \in E(G)$ is defined as $\frac{1}{\sqrt{d(u)d(v)}}$. For undefined terminology and notation we refer the reader to [1].

Aouchiche et al. [2] posed the following conjecture on the relationship between the Randić index and the diameter:

Conjecture 1 *([2]) For any connected graph G on $n \geq 3$ vertices with the Randić index $R(G)$ and the diameter $D(G)$, we have*

$$R(G) - D(G) \geq \sqrt{2} - \frac{n+1}{2} \quad and \quad \frac{R(G)}{D(G)} \geq \frac{n-3+2\sqrt{2}}{2n-2}, \tag{2}$$

with equalities if and only if G is a path, namely, $G \cong P_n$. ∎

Liu et al. [3] claimed that they proved the first part of Conjecture 1. But we find that their proof is invalid. In the following section, we will point out that the proofs of their two crucial lemmas are incorrect. Moreover, we give two counterexamples to show that the conclusions of their two lemmas are essentially wrong, which means they did not provide a proof for the first part of the conjecture.

2 On the errors in the paper

Error 1 *(Lemma 2.6 in [3]) Let w be a pendent vertex of a connected graph G and uv be the edge of maximum weight in $G - w$, then*

$$R(G) > R(G - uv) \tag{3}$$

On page 754 (line 1), the authors wrote:

If $R(G) \leq R(G - uv)$, then $R(G - uv) \geq R(G) > R(G - w) > R(G - w - uv)$, i.e. $R(G - uv) - R(G - w - uv) > R(G) - R(G - w)$, which gives

$$\begin{aligned} &\frac{1}{\sqrt{d(u)-1}} + \sum_{x \sim u, x \neq \{v,w\}} \frac{1}{\sqrt{d(x)}} \left(\frac{1}{\sqrt{d(u)-1}} - \frac{1}{\sqrt{d(u)-2}} \right) \\ &> \frac{1}{\sqrt{d(u)}} + \frac{1}{\sqrt{d(v)}} \left(\frac{1}{\sqrt{d(u)}} - \frac{1}{\sqrt{d(u)-1}} \right) \\ &\quad + \sum_{x \sim u, x \neq \{v,w\}} \frac{1}{\sqrt{d(x)}} \left(\frac{1}{\sqrt{d(u)}} - \frac{1}{\sqrt{d(u)-1}} \right), \end{aligned} \tag{4}$$

where $x \sim u$ *means that* x *is incident to* u *in* G*, which is correct. Next, they transformed the inequality (4) into the following:*

$$\begin{aligned}&\left(\frac{1}{\sqrt{d(v)}}-1\right)\left(\frac{1}{\sqrt{d(u)}}-\frac{1}{\sqrt{d(u)-1}}\right)\\&+\sum_{x\sim u, x\neq\{v,w\}}\frac{1}{\sqrt{d(x)}}\left(\frac{1}{\sqrt{d(u)}}+\frac{1}{\sqrt{d(u)-2}}-\frac{2}{\sqrt{d(u)-1}}\right)<0.\end{aligned}\tag{5}$$

Then they proved

$$\left(\frac{1}{\sqrt{d(v)}}-1\right)\left(\frac{1}{\sqrt{d(u)}}-\frac{1}{\sqrt{d(u)-1}}\right)>0\tag{6}$$

and

$$\sum_{x\sim u, x\neq\{v,w\}}\frac{1}{\sqrt{d(x)}}\left(\frac{1}{\sqrt{d(u)}}+\frac{1}{\sqrt{d(u)-2}}-\frac{2}{\sqrt{d(u)-1}}\right)>0.\tag{7}$$

So

$$\begin{aligned}&\left(\frac{1}{\sqrt{d(v)}}-1\right)\left(\frac{1}{\sqrt{d(u)}}-\frac{1}{\sqrt{d(u)-1}}\right)\\&+\sum_{x\sim u, x\neq\{v,w\}}\frac{1}{\sqrt{d(x)}}\left(\frac{1}{\sqrt{d(u)}}+\frac{1}{\sqrt{d(u)-2}}-\frac{2}{\sqrt{d(u)-1}}\right)>0,\end{aligned}\tag{8}$$

contradicting the inequality (5)*. Therefore,* $R(G) > R(G - uv)$ *and the lemma holds. However, if we check carefully the transformation from inequality* (4) *to* (5)*, we can see the inequality* (5) *is incorrect, which should be written as*

$$\begin{aligned}&\left(\frac{1}{\sqrt{d(v)}}+1\right)\left(\frac{1}{\sqrt{d(u)}}-\frac{1}{\sqrt{d(u)-1}}\right)\\&+\sum_{x\sim u, x\neq\{v,w\}}\frac{1}{\sqrt{d(x)}}\left(\frac{1}{\sqrt{d(u)}}+\frac{1}{\sqrt{d(u)-2}}-\frac{2}{\sqrt{d(u)-1}}\right)<0.\end{aligned}\tag{9}$$

Now, we cannot get the inequality (6), and thus the contradiction. In fact, Lemma 2.6 in [3] is essentially wrong. We give a counterexample (Example 1) of the lemma to illustration its incorrectness. ∎

Error 2 *(Lemma 2.7 in [3]) Let* w, z *be two pendent vertices in a connected graph* G *such that* $d(w, z) \geq 4$ *and* uv *be the edge of maximum weight in* $G - w - z$*, then*

$$R(G) > R(G - uv)\tag{10}$$

The proof of this lemma is also incorrect and the mistake is the same as the case of Lemma 2.6 in [3]. We also give a counterexample (Example 2) to show that the lemma is false. $\blacksquare$

Example 1 *Figure 1 depicts the graphs G and $G-uv$, where w is a pendent vertex of G and uv is the edge with maximum weight in $G-w$. Then*

$$R(G) - R(G-uv) = \frac{2\sqrt{3}+8-3\sqrt{2}-3\sqrt{6}}{6} < 0.$$

$\blacksquare$

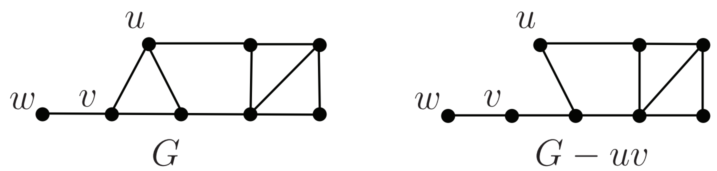

Figure 1: A counterexample of Lemma 2.6 in [3]

Example 2 *Figure 2 depicts the graphs G and $G-uv$, where w, z are two pendent vertices of G and uv is the edge with maximum weight in $G-w-z$. Then*

$$R(G) - R(G-uv) = \frac{4\sqrt{3}+4-6\sqrt{2}-\sqrt{6}}{6} < 0.$$

$\blacksquare$

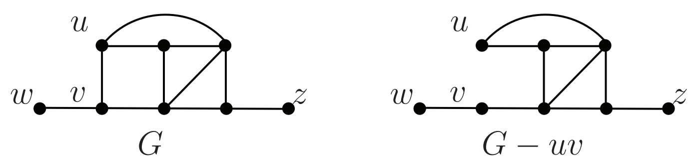

Figure 2: A counterexample of Lemma 2.7 in [3]

In the proof of Theorem 3.2 in [3], when they considered the case that $\lambda(G) = 1$ and every cut-edge of G is a pendent edge, Lemma 2.6 and Lemma

2.7 in [3] were used. Since the two lemmas are incorrect, their proof is invalid. In fact, it is not easy to overcome this case and we should find another method to prove this case or even the whole theorem.

References

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