The complexity of determining the rainbow vertex-connection of a graph*

Lily Chen, Xueliang Li, Yongtang Shi

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, China

Email: lily60612@126.com, lxl@nankai.edu.cn, shi@nankai.edu.cn

Abstract

A vertex-colored graph is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors, which was introduced by Krivelevich and Yuster. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. In this paper, we study the complexity of determining the rainbow vertex-connection of a graph and prove that computing $rvc(G)$ is NP-Hard. Moreover, we show that it is already NP-Complete to decide whether $rvc(G) = 2$. We also prove that the following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.

Keywords: coloring; rainbow vertex-connection; computational complexity

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notations and terminology of Bondy and Murty [1].

An edge-colored graph is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. The concept of rainbow connection in graphs was introduced by Chartrand et al. in [4]. The *rainbow connection number* of a connected

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graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. Observe that $diam(G) \leq rc(G) \leq n-1$, where $diam(G)$ denotes the diameter of G. It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, and that $rc(G) = n-1$ if and only if G is a tree. There are some approaches to study the bounds of $rc(G)$, for which we refer to [2, 5, 7].

As an analogous concept, Krivelevich and Yuster proposed a concept of the rainbow vertex-connection in [5]. A vertex-colored graph is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. Such a path is called a *rainbow vertex-connected path*. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. An easy observation is that if G has an order n, then $rvc(G) \leq n-2$ and $rvc(G) = 0$ if and only if G is a complete graph. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter of G is 1 or 2. For the rainbow connection and the rainbow vertex-connection, some examples are given in [5] showing that there is no upper bound for one of parameters in terms of the other. Krivelevich and Yuster [5] proved that if G is a graph with n vertices and minimum degree δ, then $rvc(G) < 11n/\delta$. The bound was then improved later, for which we refer to [6].

Besides its theoretical interest as a natural combinatorial concept, the rainbow connection can also find applications in networking. Suppose we want to route messages in a cellular network in such a way that each link on the route between two vertices is assigned with a distinct channel. The minimum number of channels that we have to use is exactly the rainbow connection of the underlying graph.

The complexity of determining the rainbow connection of a graph has been studied in literature. In [2], Caro et al. conjectured that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. In [3], Chakraborty et al. confirmed this conjecture. Motivated by the work of [3], we consider the complexity of determining the rainbow vertex-connection $rvc(G)$ of a graph. It is not hard to image that this problem is also NP-hard, but a rigorous proof is necessary. This paper is to give such a proof that computing $rvc(G)$ is NP-Hard. Our proof follows a similar idea of [3], but differently by reducing the problem of 3-SAT to some other new problems. Moreover, we show that it is already NP-Complete to decide whether $rvc(G) = 2$. We also prove that the following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.

2 The problem of rainbow vertex-connection

For two problems A and B, we write $A \preceq B$, if problem A is polynomially reducible to problem B. Now, we give our first theorem.

Theorem 1 *The following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.*

Now we define Problem 1 and Problem 2 in the following. We will prove Theorem 1 by reducing Problem 1 to Problem 2, and then the problem of 3-SAT (see [1]) to Problem 1.

Problem 1 The $s-t$ rainbow vertex-connection.

Given: Vertex-colored graph G with two specified vertices s and t.
Decide: Whether there is a rainbow vertex-connected path connecting s and t ?

Problem 2 The rainbow vertex-connection.

Given: Vertex-colored graph G.
Decide: Whether G is rainbow vertex-connected under the vertex coloring ?

Lemma 1 *Problem 1 $\preceq$ Problem 2.*

Proof. Given a vertex-colored graph G with two specified vertices s and t. We want to construct a new graph G' with a vertex coloring such that G' is rainbow vertex-connected if and only if there is a rainbow vertex-connected path connecting s and t in G.

Let $V = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be the vertices of G, where $v_1 = s$ and $v_n = t$. We construct a new graph $G' = (V', E')$ as follows. Set

$$V' = V \cup \{s', t', a, b\}$$

and

$$E' = E \cup \{s's, t't\} \cup \{av_i, bv_i : i \in [n]\}.$$

Let c be the vertex coloring of G. We define the vertex coloring c' of G' as follows: $c'(v_i) = c(v_i)$ for $i \in \{2, 3, \ldots, n-1\}$; $c'(s) = c'(a) = c_1$, $c'(t) = c'(b) = c_2$, where c_1, c_2 are two new colors; and the vertices s' and t' are assigned any colors already used.

Suppose that c' makes G' rainbow vertex-connected. Since each path Q from s' to t' in G' must go through s and t, Q cannot contain a and b as $c'(s) = c'(a) = c_1$ and $c'(t) = c'(b) = c_2$. Therefore, any rainbow vertex-connected path from s' to t' in G' must contain a rainbow vertex-connected path from s to t in G. Thus, there is a rainbow vertex-connected path connecting s and t in G under the vertex coloring c.

Now assume that there is a rainbow vertex-connected path from s to t in G under the vertex coloring c. We are ready to prove that G' is rainbow vertex-connected. First, the rainbow vertex-connected path from s' to v_i can be formed by going through s and b, then to v_i for $i \in \{2, 3, \ldots, n\}$. The rainbow vertex-connected path from s' to t' can go through s and t and along a rainbow vertex-connected path from s to t in G. The rainbow vertex-connected path from t' to v_i can be formed by going through t and a, then to v_i for $i \in \{2, 3, \ldots, n\}$. For each of the other pairs of vertices, similarly there is a path connecting them with a length less than 3. Thus, they are obviously rainbow vertex-connected. ∎

Lemma 2 *3-SAT $\preceq$ Problem 1.*

Proof. Let ϕ be an instance of the 3-SAT with clauses $c_1, c_2, \ldots, c_m$ and variables $x_1, x_2, \ldots, x_n$. We construct a graph G_ϕ with two specified vertices s and t. Let $k \geq m$ and $\ell \geq m$ be two integers.

First, we introduce k new vertices $v_1^{i,j}, v_2^{i,j}, \ldots, v_k^{i,j}$ for each $x_j \in c_i$ and ℓ new vertices $\overline{v}_1^{i,j}, \overline{v}_2^{i,j}, \ldots, \overline{v}_\ell^{i,j}$ for each $\overline{x}_j \in c_i$.

Next, for each $v_a^{i,j}$ with $a \in [k]$, where and in what follows $[k]$ denotes the set $\{1, 2, \cdots, k\}$, we introduce ℓ new vertices $v_{a1}^{i,j}, v_{a2}^{i,j}, \ldots, v_{a\ell}^{i,j}$, which form a path in this order (we call $v_{a1}^{i,j}$ and $v_{a\ell}^{i,j}$ the initial vertex and the terminal vertex of the path, respectively). Similarly, for each $\overline{v}_b^{i,j}$ with $b \in [\ell]$, we introduce k new vertices $\overline{v}_{1b}^{i,j}, \overline{v}_{2b}^{i,j}, \ldots, \overline{v}_{kb}^{i,j}$, which form a path in that order (we call $\overline{v}_{1b}^{i,j}$ and $\overline{v}_{kb}^{i,j}$ the initial vertex and the terminal vertex of the path, respectively). Therefore, for $x_j \in c_i$ there are k paths of length $\ell - 1$, and for $\overline{x}_j \in c_i$ there are ℓ paths of length $k - 1$. We use S_i to denote the set of all the paths corresponding to the three variables in c_i for $1 \leq i \leq m$, and define $S_0 = \{s\}$. For each path P in S_i ($i \in [m]$), we join the initial vertex of P to the terminal vertices of all the paths in S_{i-1}. And for each path in S_m, we join its terminal vertex to t. Thus, we obtain a new graph G_ϕ.

Now we define a vertex coloring of G_ϕ. For every variable x_j, we introduce $k \times \ell$ distinct colors $\alpha_{1,1}^j, \alpha_{1,2}^j, \ldots, \alpha_{k,\ell}^j$. For all $i \in [m]$, we color the vertices $v_{a1}^{i,j}, v_{a2}^{i,j}, \ldots, v_{a\ell}^{i,j}$ with colors $\alpha_{a,1}^j, \alpha_{a,2}^j, \ldots, \alpha_{a,\ell}^j$, respectively, and color $\overline{v}_{1b}^{i,j}, \overline{v}_{2b}^{i,j}, \ldots, \overline{v}_{kb}^{i,j}$ with colors $\alpha_{1,b}^j, \alpha_{2,b}^j, \ldots, \alpha_{k,b}^j$, respectively, where $a \in [k]$ and $b \in [\ell]$.

Now suppose that G_ϕ contains a rainbow vertex-connected path Q connecting s and t. Note that Q must contain exactly one of the newly built paths in each S_i for $i \in [m]$, and the paths $v_{a1}^{i,j} v_{a2}^{i,j} \ldots v_{a\ell}^{i,j}$ and $\overline{v}_{1b}^{i',j} \overline{v}_{2b}^{i',j} \ldots \overline{v}_{kb}^{i',j}$ cannot both appear in Q for any $i \neq i' \in [m]$

since the color $\alpha_{a,b}^{j}$ appears in both the two paths. If $v_{a1}^{i,j}v_{a2}^{i,j}\ldots v_{a\ell}^{i,j}$ appears in Q, set $x_j = 1$, and if $\overline{v}_{1b}^{i,j}\overline{v}_{2b}^{i,j}\ldots\overline{v}_{kb}^{i,j}$ appears in Q, set $x_j = 0$. Clearly, we can conclude that ϕ is a YES instance of the 3-SAT in this assignment.

On the other hand, suppose that ϕ is a YES instance of the 3-SAT, we will find a rainbow vertex-connected path connecting s and t as follows.

(1) For each $i \in [m]$, if there exists a $j \in [n]$ such that $x_j \in c_i$ and $x_j = 1$, then we choose a path Q_i as $v_{a1}^{i,j}v_{a2}^{i,j}\ldots v_{a\ell}^{i,j}$ for some $a \in [k]$ satisfying that $v_{a1}^{i',j}v_{a2}^{i',j}\ldots v_{a\ell}^{i',j}$ has not been chosen for all $i' \in [m]$. Note that we can always do this, since $k \geq m$.

(2) For each $i \in [m]$, if there exists a $j \in [n]$ such that $\overline{x}_j \in c_i$ and $x_j = 0$, then we choose a path Q_i as $\overline{v}_{1b}^{i,j}\overline{v}_{2b}^{i,j}\ldots\overline{v}_{kb}^{i,j}$ for some $b \in [\ell]$ satisfying that $\overline{v}_{1b}^{i',j}\overline{v}_{2b}^{i',j}\ldots\overline{v}_{kb}^{i',j}$ has not been chosen for all $i' \in [m]$. Similarly, since $\ell \geq m$, we can always do this.

Therefore, for each $i \in [m]$, we can choose a path Q_i, and for convenience, we denote it by $Q_i = y_{i1}y_{i2}\ldots y_{ir}$, where $r = k$ or ℓ. All these paths together with s and t form a path $Q = sy_{11}\ldots y_{1r}y_{21}\ldots y_{2r}\ldots y_{m1}\ldots y_{mr}t$ connecting s and t by the construction of the graph G_ϕ. We can conclude that Q is a rainbow vertex-connected path under the coloring of G_ϕ. ∎

3 The problem of rainbow vertex-connection 2

Before proceeding, we first define the following three problems.

Problem 3 The rainbow vertex-connection 2.

Given: Graph $G = (V, E)$.
Decide: Whether there is a vertex coloring of G with two colors such that all pairs $(u, v) \in V(G) \times V(G)$ are rainbow vertex-connected ?

Problem 4 The subset rainbow vertex-connection 2.

Given: Graph $G = (V, E)$ and a set of pairs $P \subseteq V(G) \times V(G)$.
Decide: Whether there is a vertex coloring of G with two colors such that all pairs $(u, v) \in P$ are rainbow vertex-connected ?

Problem 5 The different subsets rainbow vertex-connection 2.

Given: Graph $G = (V, E)$ and two disjoint subsets V_1, V_2 of V with a one to one correspondence $f : V_1 \to V_2$.
Decide: Whether there is a vertex coloring of G with two colors such that G is rainbow vertex-connected and for each $v \in V_1$, v and $f(v)$ are assigned different colors.

In the following, we will reduce Problem 4 to Problem 3, and then Problem 5 to Problem 4. Finally, we will show that it is NP-Complete to decide whether $rvc(G) = 2$ by reducing the 3-SAT to Problem 5.

Lemma 3 *Problem 4 $\preceq$ Problem 3.*

Proof. Given a graph $G = (V, E)$ and a set of pairs $P \subseteq V(G) \times V(G)$, we construct a new graph $G' = (V', E')$ as follows.

For each vertex $v \in V$, we introduce a new vertex x_v; for every pair $(u, v) \in (V \times V) \setminus P$, we introduce two new vertices $x^1_{(u,v)}$ and $x^2_{(u,v)}$; we also add two new vertices s, t. Set

$$V' = V \cup \{x_v : v \in V\} \cup \{x^1_{(u,v)}, x^2_{(u,v)} : (u, v) \in (V \times V) \setminus P\} \cup \{s, t\}$$

and
$E' = E \cup \{vx_v : v \in V\} \cup \{ux^1_{(u,v)}, x^1_{(u,v)}x^2_{(u,v)}, x^2_{(u,v)}v : (u, v) \in (V \times V) \setminus P\} \cup \{sx^1_{(u,v)}, tx^2_{(u,v)} : (u, v) \in (V \times V) \setminus P\} \cup \{sx_v, tx_v : v \in V\}$.

In the following, we will prove that G' is rainbow vertex-connected with two colors if and only if there is a vertex coloring of G with two colors such that all pairs $(u, v) \in P$ are rainbow vertex-connected.

Now suppose that there is a vertex coloring of G' with two colors which makes G' rainbow vertex-connected. For each pair $(u, v) \in P$, by the construction of G', the paths connecting u and v with lengths at most 3 have to be in G. Observe that G is a subgraph of G'. Thus, considering the restriction of the coloring of G' on G, all pairs in P are rainbow vertex-connected.

On the other hand, let $c : V \to \{1, 2\}$ be a vertex coloring of G such that all pairs $(u, v) \in P$ are rainbow vertex-connected. We extend the coloring as follows: $c(x_v) = 1$ for all $v \in V$; $c(x^1_{(u,v)}) = 1$ and $c(x^2_{(u,v)}) = 2$ for all $(u, v) \in (V \times V) \setminus P$; $c(s) = c(t) = 2$. Now we show that G' is indeed rainbow vertex-connected under this vertex coloring. Let u and v be any two vertices in G'. We consider the following cases.

(1) $(u, v) \in P$. There is a rainbow vertex-connected path connecting u and v by the assumption.

(2) $(u, v) \in (V \times V) \setminus P$. In this case, $ux^1_{(u,v)}x^2_{(u,v)}v$ is a rainbow vertex-connected path.

(3) $u \in V(G)$ and $v = x_w$. If $u \neq w$, then ux_utv is a rainbow vertex-connected path; otherwise, uv is an edge of G'.

(4) $u \in V(G)$ and $v = x^j_{(y,w)}$, where $j = 1, 2$. In this case, ux_usv is a rainbow vertex-connected path if $j = 1$, and ux_utv is a rainbow vertex-connected path if $j = 2$.

(5) $u \in V(G)$ and $v = s$ or t. In this case, ux_uv is a rainbow vertex-connected path.

(6) $u = x^1_{(y,w)}$ and $v = x^2_{(y',w')}$. In this case, $usx^1_{(y',w')}v$ is a rainbow vertex-connected path.

(7) For the other cases of u and v, there is a rainbow vertex-connected path connecting u and v since the distance of u and v in G' is at most 2. ∎

Lemma 4 *Problem 5 $\preceq$ Problem 4.*

Proof. Given a graph $G = (V, E)$ and two disjoint subsets V_1, V_2 of V with a one to one correspondence f. Assume that $V_1 = \{v_1, v_2, \dots, v_k\}$ and $V_2 = \{w_1, w_2, \dots, w_k\}$ satisfying $w_i = f(v_i)$ for each $i \in [k]$. We construct a new graph $G' = (V', E')$ as follows.

We introduce six new vertices $x^1_{v_iw_i}, x^2_{v_iw_i}, x^3_{v_iw_i}, x^4_{v_iw_i}, x^5_{v_iw_i}, x^6_{v_iw_i}$ for each pair (v_i, w_i), where $i \in [k]$; we add a new vertex s. Set

$$V' = V \cup \{x^j_{v_iw_i} : i \in [k], j \in [6]\} \cup \{s\},$$

and
$E' = E \cup \{sx^5_{v_iw_i}, x^5_{v_iw_i}v_i, v_ix^1_{v_iw_i}, x^1_{v_iw_i}x^2_{v_iw_i}, x^2_{v_iw_i}x^3_{v_iw_i}, x^3_{v_iw_i}x^4_{v_iw_i}, x^4_{v_iw_i}w_i,\ w_ix^6_{v_iw_i}, x^6_{v_iw_i}s : i \in [k]\}$.

We define P by
$P = \{(u, v) : u, v \in V\} \cup \{(x^5_{v_iw_i}, x^2_{v_iw_i}), (v_i, x^3_{v_iw_i}), (x^1_{v_iw_i}, x^4_{v_iw_i}), (x^2_{v_iw_i}, w_i), (x^3_{v_iw_i}, x^6_{v_iw_i}) : i \in [k]\}$.

Suppose that there is a vertex coloring c of G' with two colors such that all pairs $(u, v) \in P$ are rainbow vertex-connected. At first, we will show that G is rainbow vertex-connected. Let u and v be any two vertices in G. We prove the following claim.

Claim 1 The path connecting u and v in G' with a length at most 3 must belong to G.

Proof. If one of u and v does not belong to $V_1 \cup V_2$, then the claim holds, obviously. Now we suppose $u, v \in V_1 \cup V_2$.

Case 1 $u = v_i$ and $v = w_j$. If $j = i$, then the shortest path connecting u and v in G' which does not belong to G is $ux^5_{uv}sx^6_{uv}v$, whose length is greater than 3. If $j \neq i$, then the shortest path connecting u and v in G' which does not belong to G is $ux^5_{uw_i}sx^6_{v_jv}v$, whose length is greater than 3.

Case 2 $u = v_i$ and $v = v_j$. In this case, the shortest path connecting u and v in G' which does not belong to G is $ux^5_{uw_i}sx^6_{vw_j}v$, whose length is greater than 3.

Case 3 $u = w_i$ and $v = w_j$. The proof of this case is similar to that of *Case 2*.

Therefore, the proof of Claim 1 is complete. ∎

Observe that G is a subgraph of G'. Consider the restriction of the vertex coloring of G' on G. Since $(u, v) \in P$ and all pairs $(u, v) \in P$ are rainbow vertex-connected, we can deduce that there exists a rainbow vertex-connected path connecting u and v in G from Claim 1. Thus, we have proved that G is rainbow vertex-connected. Now we prove $c(v_i) \neq c(w_i)$ for any $i \in [k]$. Since $(x^5_{v_iw_i}, x^2_{v_iw_i}) \in P$ are rainbow vertex-connected in G' and the only path between them with a length at most 3 is $x^5_{v_iw_i}v_ix^1_{v_iw_i}x^2_{v_iw_i}$, we have $c(v_i) \neq c(x^1_{v_iw_i})$. Similarly, the fact that $(v_i, x^3_{v_iw_i}), (x^1_{v_iw_i}, x^4_{v_iw_i}), (x^2_{v_iw_i}, w_i), (x^3_{v_iw_i}, x^6_{v_iw_i}) \in P$ are rainbow vertex-connected in G' implies that $c(x^1_{v_iw_i}) \neq c(x^2_{v_iw_i})$, $c(x^2_{v_iw_i}) \neq c(x^3_{v_iw_i})$, $c(x^3_{v_iw_i}) \neq c(x^4_{v_iw_i})$, $c(x^4_{v_iw_i}) \neq c(w_i)$, respectively. Therefore, we can observe that $c(v_i) = c(x^2_{v_iw_i}) = c(x^4_{v_iw_i})$ and $c(w_i) = c(x^3_{v_iw_i}) = c(x^1_{v_iw_i})$, which implies $c(v_i) \neq c(w_i)$ as $c(x^2_{v_iw_i}) \neq c(x^3_{v_iw_i})$.

On the other hand, suppose that there is a vertex coloring c of G with two colors such that G is rainbow vertex-connected and v_i, w_i are colored differently for any $i \in [k]$. We color G' with a vertex coloring c' as follows: $c'(v) = c(v)$ for $v \in V$; if $c(v_i) = 1$ and $c(w_i) = 2$, then $c'(x^1_{v_iw_i}) = c'(x^3_{v_iw_i}) = 2$ and $c'(x^2_{v_iw_i}) = c'(x^4_{v_iw_i}) = 1$; if $c(v_i) = 2$ and $c(w_i) = 1$, then $c'(x^1_{v_iw_i}) = c'(x^3_{v_iw_i}) = 1$ and $c'(x^2_{v_iw_i}) = c'(x^4_{v_iw_i}) = 2$; for any other vertex u in G', $c'(u) = 1$ or 2 arbitrarily. Now we check that all $(u, v) \in P$ are rainbow vertex-connected. By the definition of P, we only need to consider the pairs $(x^5_{v_iw_i}, x^2_{v_iw_i}), (v_i, x^3_{v_iw_i}), (x^1_{v_iw_i}, x^4_{v_iw_i}), (x^2_{v_iw_i}, w_i), (x^3_{v_iw_i}, x^6_{v_iw_i})$ for $i \in [k]$, since G is rainbow vertex-connected. Notice that under the vertex coloring c' of G', the paths $x^5_{v_iw_i}v_ix^1_{v_iw_i}x^2_{v_iw_i}$, $v_ix^1_{v_iw_i}x^2_{v_iw_i}x^3_{v_iw_i}$, $x^1_{v_iw_i}x^2_{v_iw_i}x^3_{v_iw_i}x^4_{v_iw_i}$, $x^2_{v_iw_i}x^3_{v_iw_i}x^4_{v_iw_i}w_i$ and $x^3_{v_iw_i}x^4_{v_iw_i}w_ix^6_{v_iw_i}$ are rainbow vertex-connected, respectively.

The proof is thus complete. ∎

Lemma 5 *3-SAT $\preceq$ Problem 5.*

Proof. Let ϕ be an instance of the 3-SAT with clauses $c_1, c_2, \ldots, c_m$ and variables $x_1, x_2, \ldots, x_n$. We construct a new graph $G_\phi = (V_\phi, E_\phi)$ and define two disjoint vertex sets with a one to one correspondence f, as follows. Add two new vertices s and t. Set

$$V_\phi = \{c_i : i \in [m]\} \cup \{x_i, \overline{x}_i : i \in [n]\} \cup \{s, t\}$$

and

$E_\phi = \{c_ic_j : i, j \in [m]\} \cup \{tx_i, t\overline{x}_i : i \in [n]\} \cup \{x_ic_j : x_i \in c_j\} \cup \{\overline{x}_ic_j : \overline{x}_i \in c_j\} \cup \{st\}$.

We define $V_1 = \{x_1, x_2, \ldots, x_n\}$, $V_2 = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$ and $f : V_1 \to V_2$ satisfying $f(x_i) = \overline{x}_i$. Now we show that G_ϕ is rainbow vertex-connected with 2 colors and x_i and $\overline{x}_i$ are assigned different colors for each $i \in [n]$ if and only if ϕ is satisfiable.

Suppose that there is a vertex coloring $c : V_\phi \rightarrow \{0, 1\}$ such that G_ϕ is rainbow vertex-connected and x_i, $\overline{x}_i$ are colored differently. We first suppose $c(t) = 0$, and set the value of x_i as the corresponding color of x_i. For each i, consider the rainbow vertex-connected path Q between the vertices s and c_i. There must exist some j such that we can write $Q = stx_jc_i$ or $Q = st\overline{x}_jc_i$. Without loss of generality, suppose $Q = stx_jc_i$. Since $c(t) = 0$, we have $c(x_j) = 1$. Thus, the value of x_j is 1, which implies $c_i = 1$ as $x_j \in c_i$ by the construction of G_ϕ. For the other case, i.e., $c(t) = 1$, we set $x_i = 1$ if $c(x_i) = 0$ and $x_i = 0$ otherwise. By some similar discussions, we can also deduce that ϕ is a YES instance of the 3-SAT.

On the other hand, for a given truth assignment of ϕ, we color G_ϕ as follows: $c(t) = 0$ and $c(c_i) = 1$ for $i \in [m]$; if $x_i = 1$, then $c(x_i) = 1$ and $c(\overline{x}_i) = 0$; otherwise, $c(x_i) = 0$ and $c(\overline{x}_i) = 1$; $c(s) = 0$ or 1 arbitrarily. Hence, by the definition of V_1 and V_2, we know that for any $u \in V_1$, u and $f(u)$ are colored differently. In the following, we will check that the graph G_ϕ is rainbow vertex-connected. Let u and v be any two vertices of G_ϕ. We only need to consider the case that $u = s$ and $v = c_i$ for any $i \in [m]$, since for all the other cases, the length of the shortest paths connecting u and v is at most 2. If $x_j \in c_i$ and $x_j = 1$, then stx_jc_i is the path required. If $\overline{x}_j \in c_i$ and $x_j = 0$, then $st\overline{x}_jc_i$ is the path required. ∎

From the above three lemmas, we can get our second theorem.

Theorem 2 *Given a graph G, deciding whether $rvc(G) = 2$ is NP-Complete. Thus, computing $rvc(G)$ is NP-Hard.*

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