SYMMETRIC IDENTITIES ON BERNOULLI POLYNOMIALS

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ABSTRACT. In this paper, we obtain a generalization of an identity due to Carlitz on Bernoulli polynomials. Then we use this generalized formula to derive two symmetric identities which reduce to some known identities on Bernoulli polynomials and Bernoulli numbers, including the Miki identity.

1. INTRODUCTION

The Bernoulli polynomials $B_n(x)$, $n = 1, 2, \dots$ are given by the generating function:

$$\sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} = \frac{te^{xt}}{e^t - 1}.$$

In particular, we call $B_n := B_n(0)$ the n-th Bernoulli number.

In [8], Miki discovered a curious identity involving Bernoulli numbers:

$$\sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k(n-k)} = \sum_{k=2}^{n-2} \binom{n}{k} \frac{B_k B_{n-k}}{k(n-k)} + 2H_n \frac{B_n}{n} \tag{1.1}$$

for $n \geqslant 4$, where

$$H_n := \sum_{i=1}^{n} \frac{1}{i}$$

is the n-th harmonic number. Later several different proofs of (1.1) were found by Shirantani and Yokoyama [10], Gessel [5], Dunne and Schubert [3]. Furthermore, in the same paper Dunne and Schubert also proved a similar identity conjectured by Matiyasevich [7]:

$$(n+2) \sum_{k=2}^{n-2} B_k B_{n-k} = 2 \sum_{k=2}^{n-2} \binom{n+2}{k} B_k B_{n-k} + n(n+1)B_n \tag{1.2}$$

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for $n \geqslant 4$. On the other hand, using a new difference-differential method, Pan and Sun [9] established the following generalizations of (1.1) and (1.2) for Bernoulli polynomials:

$$\sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(y)}{k(n-k)} - \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{B_k(x-y)B_{n-k}(y) + B_k(y-x)B_{n-k}(x)}{k^2}$$
$$= H_{n-1} \frac{B_n(x) + B_n(y)}{n} + \frac{B_n(x) - B_n(y)}{n(x-y)}, \tag{1.3}$$

and

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(y) - \sum_{k=0}^{n} \binom{n+1}{k+1} \frac{B_k(x-y)B_{n-k}(y) + B_k(y-x)B_{n-k}(x)}{k+2}$$
$$= \frac{B_{n+1}(x) + B_{n+1}(y)}{(x-y)^2} - \frac{2}{n+2} \cdot \frac{B_{n+2}(x) - B_{n+2}(y)}{(x-y)^3}, \tag{1.4}$$

for $n \geqslant 2$.

With the help of some symmetric identities on Bernoulli polynomials given in [12], they also proved a polynomial-type extension of an identity due to Woodock [14]:

$$A_{m-1,n}(x) = A_{m,n-1}(x) \tag{1.5}$$

for positive integers m, n, where

$$A_{m,n}(x) = \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k B_{m+k}(x) B_{n-k}(2x) - \frac{1}{n} B_m(x) B_n(x).$$

Subsequently, Sun and Pan [13] discovered the following symmetric identity as a generalization of the above identities:

$$r \begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s \begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t \begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0 \tag{1.6}$$

provided that $r + s + t = n$ and $x + y + z = 1$, where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n = \sum_{k=0}^{n} (-1)^k \binom{s}{k} \binom{t}{n-k} B_{n-k}(x) B_k(y).$$

Motivated by the results of Dilcher [2], the referee of [13] asked whether there exists a generalization of (1.6) involving sums of products of more Bernolli polynomials. In this paper, we shall give such a generalization.

Theorem 1.1. *Let m and n be positive integers. Suppose that $r_1,\ldots,r_m$ are arbitrary complex numbers. Then*

$$r_{m+1}\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n}}\prod_{j=1}^{m}\binom{r_j}{k_j}B_{k_j}(x_j)$$
$$=-\sum_{i=1}^{m}r_i\sum_{\substack{k_1,\ldots,k_m\geqslant 0\\ k_1+\cdots+k_m=n}}\binom{r_{m+1}}{k_i}B_{k_i}(1-x_i)\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\binom{r_j}{k_j}B_{k_j}(x_j-x_i+\mathbf{1}_{j>i}),\qquad(1.7)$$

where $r_{m+1}=n-r_1-\cdots-r_m$ and $\mathbf{1}_{j>i}=1$ or 0 according to whether $j>i$.

Let us explain why (1.7) is equivalent to (1.6) when $m=2$. It is not difficult to check that $B_k(1-x)=(-1)^kB_k(x)$ (cf. [4, pp.102-103]). Hence in view of (1.7),

$$\begin{aligned}
&r\sum_{k=0}^{n}\binom{s}{k}B_k(1-y)\binom{t}{n-k}B_{n-k}(x)\\
=\;&-s\sum_{k=0}^{n}\binom{r}{k}B_k(1-(1-y))\binom{t}{n-k}B_{n-k}(x-(1-y)+1)\\
&-t\sum_{k=0}^{n}\binom{r}{k}B_k(1-x)\binom{s}{n-k}B_{n-k}((1-y)-x)\\
=\;&-s\sum_{k=0}^{n}(-1)^k\binom{t}{k}B_k(1-x-y)\binom{r}{n-k}B_{n-k}(y)\\
&-t\sum_{k=0}^{n}(-1)^k\binom{r}{k}B_k(x)\binom{s}{n-k}B_{n-k}(1-x-y),
\end{aligned}$$

which is indeed (1.6) by setting $z=1-x-y$.

2. Proof of Theorem 1.1

For a power series $f(t_1,\ldots,t_m)$, let $[t_1^{n_1}\cdots t_m^{n_m}]f(t_1,\ldots,t_m)$ denote the coefficient of $t_1^{n_1}\cdots t_m^{n_m}$ in $f(t_1,\ldots,t_m)$. The following lemma is a generalization of an identity due to Carlitz [1, Eq. (7)]:

Lemma 2.1.

$$\sum_{i=1}^{m}n_iB_{n_i-1}(x_i)\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}B_{n_j}(x_j)$$
$$=\sum_{i=1}^{m}n_i\sum_{\substack{k_1,\ldots,k_m\geqslant 0\\ k_1+\cdots+k_m=n_1+\cdots+n_m}}B_{k_i-1}(x_i)\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\binom{n_j}{k_j}B_{k_j}(x_j-x_i+\mathbf{1}_{j>i}),$$

where we set $B_n(x)=0$ for $n<0$.

Proof. Consider

$$
\begin{aligned}
&(t_1+\cdots+t_m)\prod_{j=1}^{m}\frac{t_je^{x_jt_j}}{e^{t_j}-1}\\
&= \frac{(t_1+\cdots+t_m)}{e^{t_1+\cdots+t_m}-1}\left(\sum_{i=1}^{m}(e^{t_i}-1)e^{\sum_{i<j\leqslant m}t_j}\right)\prod_{j=1}^{m}\frac{t_je^{x_jt_j}}{e^{t_j}-1}\\
&= \sum_{i=1}^{m}\frac{t_i(t_1+\cdots+t_m)e^{x_i(t_1+\cdots+t_m)}}{e^{t_1+\cdots+t_m}-1}\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\frac{t_je^{(x_j-x_i+\mathbf{1}_{j>i})t_j}}{e^{t_j}-1}. \qquad (2.1)
\end{aligned}
$$

Clearly, we have

$$
[t_1^{n_1}\cdots t_m^{n_m}](t_1+\cdots+t_m)\prod_{j=1}^{m}\frac{t_je^{x_jt_j}}{e^{t_j}-1}=\sum_{i=1}^{m}\frac{B_{n_i-1}(x_i)}{(n_i-1)!}\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\frac{B_{n_j}(x_j)}{n_j!}.
$$

Now, for each $1\leqslant i\leqslant m$,

$$
\begin{aligned}
&[t_1^{n_1}\cdots t_m^{n_m}]\frac{t_i(t_1+\cdots+t_m)e^{x_i(t_1+\cdots+t_m)}}{e^{t_1+\cdots+t_m}-1}\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\frac{t_je^{(x_j-x_i+\mathbf{1}_{j>i})t_j}}{e^{t_j}-1}\\
&= \sum_{k_1,\cdots,k_{i-1},k_{i+1},\cdots,k_m\geqslant 0}\frac{B_{k_1+\ldots+k_{i-1}+k_{i+1}+\ldots+k_m+n_i-1}(x_i)}{k_1!\cdots k_{i-1}!(n_i-1)!k_{i+1}!\cdots k_m!}\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\frac{B_{n_j-k_j}(x_j-x_i+\mathbf{1}_{j>i})}{(n_j-k_j)!}.
\end{aligned}
$$

Equating the coefficients of $t_1^{n_1}\cdots t_m^{n_m}$ on both sides of (2.1) gives the desired identity. $\square$

Proof of Theorem 1.1. Applying Lemma 2.1, we have

$$
\begin{aligned}
&(n-r_1-\cdots-r_m)\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n}}\prod_{j=1}^m\binom{r_j}{k_j}B_{k_j}(x_j)\\
=&-\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n}}\sum_{i=1}^m(k_i+1)\binom{r_i}{k_i+1}B_{k_i}(x_i)\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\binom{r_j}{k_j}B_{k_j}(x_j)\\
=&-\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n+1}}\left(\sum_{i=1}^m k_iB_{k_i-1}(x_i)\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}B_{k_j}(x_j)\right)\prod_{j=1}^m\binom{r_j}{k_j}\\
=&-\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n+1}}\left(\sum_{i=1}^m k_i\sum_{\substack{l_1,\ldots,l_m\geqslant 0\\ l_1+\cdots+l_m=n}}B_{l_i}(x_i)\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\binom{k_j}{l_j}B_{l_j}(x_j-x_i+\mathbf{1}_{j>i})\right)\prod_{j=1}^m\binom{r_j}{k_j}\\
=&-\sum_{i=1}^m r_i\sum_{\substack{l_1,\ldots,l_m\geqslant 0\\ l_1+\cdots+l_m=n}}B_{l_i}(x_i)\Bigg(\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n+1}}\binom{r_i-1}{k_i-1}\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\binom{r_j-l_j}{k_j-l_j}\Bigg)\\
&\cdot\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\binom{r_j}{l_j}B_{l_j}(x_j-x_i+\mathbf{1}_{j>i}).
\end{aligned}
$$

By the Chu-Vandermonde identity (cf. [6, Eq. (5.27)]), we have

$$
\begin{aligned}
&\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n+1}}\binom{r_i-1}{k_i-1}\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\binom{r_j-l_j}{k_j-l_j}\\
=&\binom{r_1+\cdots+r_m-1-l_1-\cdots-l_{i-1}-l_{i+1}-\cdots-l_m}{k_1+\cdots+k_m-1-l_1-\cdots-l_{i-1}-l_{i+1}-\cdots-l_m}\\
=&\binom{r_1+\cdots+r_m-1-n+l_i}{l_i}\\
=&(-1)^{l_i}\binom{n-r_1-\cdots-r_m}{l_i}.
\end{aligned}
$$

This completes the proof. $\square$

3. A generalization of Dunne and Schubert's identity

In [3], Dunne and Schubert also proposed an extension of Miki's identity (1.1) involving the gamma function $\Gamma(z)$:

$$\frac{1}{\Gamma(2p+2n)}\sum_{k=1}^{n-1} B_{2k}B_{2n-2k}\frac{\Gamma(p+2k)\Gamma(p+2n-2k)}{\Gamma(2k+1)\Gamma(2n-2k+1)}$$

$$=2\Gamma(p+1)\sum_{k=1}^{n}\frac{B_{2k}B_{2n-2k}}{(2k)!(2n-2k)!}\frac{\Gamma(p+2k)\Gamma(2p+2n-1)}{\Gamma(2p+2k+1)}+\frac{2B_{2n}}{(2n)!}\sum_{k=1}^{2n-1}\beta(p+k,p+1),$$

where $\beta(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. However, applying Lemma 2.1 and the identity

$$\frac{\Gamma(p+k)}{\Gamma(k+1)}=(-1)^k\Gamma(p)\binom{-p}{k}$$

for $p\notin\{0,-1,-2,\ldots\}$, we are led to the following generalization.

Theorem 3.1. *Let m and n be positive integers. Suppose that $p_1,\ldots,p_m$ are non-integral complex numbers. Then*

$$\Gamma(p_{m+1}+1)\sum_{\substack{k_1,\cdots,k_m\geqslant 0\\ k_1+\cdots+k_m=n}}\prod_{j=1}^{m}\frac{\Gamma(p_j+k_j)}{\Gamma(k_j+1)}B_{k_j}(x_j)$$

$$=-\sum_{i=1}^{m}\Gamma(p_i+1)\sum_{\substack{k_1,\ldots,k_m\geqslant 0\\ k_1+\cdots+k_m=n}}\frac{\Gamma(p_{m+1}+k_i)}{\Gamma(k_i+1)}B_{k_i}(1-x_i)\prod_{\substack{1\leqslant j\leqslant m\\ j\neq i}}\frac{\Gamma(p_j+k_j)}{\Gamma(k_j+1)}B_{k_j}(x_j-x_i+\mathbf{1}_{j>i}),\tag{3.2}$$

where $p_{m+1}=-(p_1+\cdots+p_m+n)$.

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