



The matching energy of random graphs[☆]

Xiaolin Chen^a, Xueliang Li^{a,*}, Huishu Lian^b

^a Center for Combinatorics and LPMC-TJKLC, Nankai University, Tianjin 300071, China

^b College of Science, China University of Mining and Technology, Xuzhou 221116, China

ARTICLE INFO

Article history:

Received 27 December 2014

Received in revised form 21 April 2015

Accepted 21 April 2015

Available online 14 May 2015

Keywords:

Matching energy

Matching polynomial

Random graph

Empirical matching distribution

ABSTRACT

The matching energy of a graph was introduced by Gutman and Wagner, which is defined as the sum of the absolute values of the roots of the matching polynomial of the graph. For the random graph $G_{n,p}$ of order n with fixed probability $p \in (0, 1)$, Gutman and Wagner (2012) proposed a conjecture that the expectation of the matching energy of $G_{n,p}$ is asymptotically equal to $\frac{8\sqrt{p}}{3\pi} n^{\frac{3}{2}}$. In this paper, using analytical tools, we confirm this conjecture by obtaining a stronger result that the matching energy of $G_{n,p}$ is asymptotically almost surely equal to $\frac{8\sqrt{p}}{3\pi} n^{\frac{3}{2}}$.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

Let G be a finite simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. A matching of G is a set of independent edges in G , and an r -matching of G is a matching of G that has exactly r edges. By $m_r(G)$ we denote the number of r -matchings in G . It is easy to verify that for $r < 0$ and $r > \lceil n/2 \rceil$, $m_r(G) = 0$. And when $r = 1$, $m_1(G)$ is the size of G . For convenience, we define $m_0(G) = 1$. The matching polynomial $m(G, x)$ [9,11,15] of a graph G is defined as

$$m(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) x^{n-2k}.$$

The matching polynomial has been widely studied and many results on the properties of the matching roots have been obtained; see [9–12,15,18]. For any graph G , all the matching roots are real. If λ is a matching root, then $-\lambda$ is also a matching root. That is, the matching roots are symmetric. Moreover, the matching polynomial has many important implications in statistical physics and chemistry; see [13,16,18].

In [17], Gutman and Wagner introduced the matching energy (ME) of a graph G , which is defined as the sum of the absolute values of the roots of the matching polynomial of G . Note that the concept of the energy $E(G)$ of a simple undirected graph G was introduced by Gutman in [14]. Afterwards, there have been lots of research papers on this topic. A systematic study of this topic can be found in the book [20]. In [17], Gutman and Wagner pointed out that the matching energy is a quantity of relevance for chemical applications. Moreover, they arrived at the simple relation

$$TRE(G) = E(G) - ME(G),$$

[☆] Supported by NSFC No. 11371205, the “973” program No. 2013CB834204, and PCSIRT.

* Corresponding author.

E-mail addresses: chxl@nankai.edu.cn (X. Chen), lxl@nankai.edu.cn (X. Li), lhs6803@126.com (H. Lian).

where $TRE(G)$ is the so-called “topological resonance energy” of the graph G . For more information about the applications of the matching energy, we refer the reader to [13,16]. Recently, there have been some results on the extremal values of the matching energy of graphs; see [5,19,21,22].

When we add an edge to a graph, the matching energy increases strictly.

Lemma 1.1 ([17]). *Let G be a graph and e one of its edges. Let $G - e$ be the subgraph obtained by deleting from G the edge e , but keeping all the vertices of G . Then*

$$ME(G - e) < ME(G).$$

Therefore, the complete graph K_n attains the maximum matching energy among all graphs of order n . In [17], Gutman and Wagner got the following lemma which gives an asymptotic estimation of the matching energy of K_n .

Lemma 1.2 ([17]). *The matching energy of the complete graph K_n is asymptotically equal to $\frac{8}{3\pi}n^{3/2}$. More precisely,*

$$ME(K_n) = \frac{8}{3\pi}n^{3/2} + O(n). \quad (1.1)$$

The above lemma can be thought as the upper bound of the matching energy of all graphs of order n . Moreover, they studied the lower bound of the matching energy of random graphs. Now we recall some notion in probability, we say an event holds almost surely (a.s.) if it occurs with probability 1. An event holds asymptotically almost surely (a.a.s.) if the probability of success goes to 1 as $n \rightarrow \infty$.

Lemma 1.3 ([17]). *Consider the random graph $G_{n,p}$ of order n with fixed probability $p \in (0, 1)$. Then*

$$ME(G_{n,p}) \geq \frac{\sqrt{p}}{\pi}n^{3/2} + O(\sqrt{n} \ln n) \quad (1.2)$$

holds asymptotically almost surely.

Based on the above analysis, they conjectured that

Conjecture 1.4 ([17]). *For any fixed probability $p \in (0, 1)$,*

$$n^{-3/2}E(ME(G_{n,p})) \longrightarrow \frac{8\sqrt{p}}{3\pi}$$

asymptotically almost surely, where $E(ME(G_{n,p}))$ is the expectation of $ME(G_{n,p})$.

This paper is to confirm the conjecture. The rest of the paper is organized as follows. In Section 2, we introduce the empirical matching distribution and list our main results: the empirical matching distribution converges weakly to the semicircle distribution; the asymptotic formula of the matching energy of random graphs. The explicit proofs will be shown in Sections 3 and 4. Throughout the paper we use the following standard asymptotic notation: as $n \rightarrow \infty$, $f(n) = o(g(n))$ means that $f(n)/g(n) \rightarrow 0$; $f(n) = O(g(n))$ means that there exists a constant C such that $|f(n)| \leq Cg(n)$.

2. Matching energy of random graphs

In this section, we present our main results of this paper.

Definition 2.1. For the random graph $G_{n,p}$ of order n with fixed probability $p \in (0, 1)$, let $x_1(G_{n,p}) \geq \dots \geq x_n(G_{n,p})$ be the roots of the matching polynomial $m(G_{n,p}, x)$, since all roots of the matching polynomial are real. Then let $\lambda_i(G_{n,p}) = \frac{1}{\sqrt{np}}x_i(G_{n,p})$ for all $1 \leq i \leq n$.

We define the empirical matching distribution (EMD) as a distribution function $F_n(x)$ where

$$F_n(x) = \frac{1}{n} \left| \{ \lambda_i(G_{n,p}) \mid \lambda_i(G_{n,p}) \leq x, i = 1, 2, \dots, n \} \right|.$$

The empirical matching distribution can be thought as the root distribution of the matching polynomial. Most work on the root distribution focuses on the spectral distributions of random matrices. The study can be traced back to the pioneer work-semicircle law discovered by Wigner in [25]. Afterwards, the research about the spectral distributions of many sorts of random matrices became the topics in mathematics and physics. For more details, we refer the reader to books [1,2,24]. In this paper, we find that the empirical matching distribution has the similar convergent property to the spectral distribution.

Theorem 2.2. For the random graph $G_{n,p}$ of order n with $p \in (0, 1)$, the empirical matching distribution $F_n(x)$ converges weakly almost surely to the standard semicircle distribution $F_{sc}(x)$, whose density is given by

$$\rho_{sc}(x) := \frac{1}{2\pi} \sqrt{4 - x^2} \quad |x| \leq 2.$$

That is, for any bounded continuous function f on $\mathbb{R}$,

$$\int_{\mathbb{R}} f dF_n(x) \longrightarrow \int_{\mathbb{R}} f dF_{sc}(x) \quad (2.1)$$

almost surely.

From the above theorem, we can estimate the matching energy of the random graph $G_{n,p}$. Before proceeding, we should note that the semicircle law has been used to study many energies, such as the energy in [8,23], the Laplacian energy in [6], the skew energy in [4], and other energies in [7].

We prove [Conjecture 1.4](#) by the following even stronger theorem.

Theorem 2.3. For $p \in (0, 1)$, the matching energy $ME(G_{n,p})$ of the random graph $G_{n,p}$ enjoys asymptotically almost surely the following equation:

$$ME(G_{n,p}) = n^{3/2} p^{1/2} \left(\frac{8}{3\pi} + o(1) \right).$$

3. Empirical matching distributions (EMDs) of random graphs

In order to prove the empirical matching distribution (EMD) of $G_{n,p}$ converges weakly to the standard semicircle distribution, we utilize the so-called moment method. The moment method has been used extensively in random matrices, specially in semicircle distribution [1,2,24]. The key point of moment method is to show that the moments of EMDs converge almost surely to the moments of the semicircle law. Thus, in this section, we just need to verify the almost sure convergence of the moments of the matching roots. For the gap between [Theorem 2.2](#) and the moment method, we can fill it with the method used in random matrices ([1], P. 11). Thus, we do not show it in this section. However, for convenience of the reader, we present it in the [Appendix](#).

In [Section 3.1](#), we review the relationship between the tree-like closed walks and the moments of matching roots. In [Section 3.2](#), we prove a weaker convergence named convergence in expectation. In [Section 3.3](#), using some probabilistic inequalities, we upgrade the convergence in expectation to the almost sure convergence.

3.1. Tree-like walks of graphs

In [10], Godsil introduced a new type of closed walks, namely the tree-like walks, which have very closed relation to the matching roots. Before proceeding, we recall some notation. A closed walk is called minimal if only the first vertex and the last vertex coincide. For any closed walk w with length nonzero, we can uniquely decompose it into the form $\alpha\beta\gamma$, where α is a path, β is a minimal closed walk and has no common vertex with α . We do this composition by finding the first minimal closed walk from the initial vertex along the walk. Then the first minimal closed walk is just β . Suppose the first vertex in β is u_0 . By deleting the closed walk β , we get a new closed walk $w' = \alpha u_0 \gamma$. We may again decompose the new closed walk w' and get the first minimal closed walk in w' . By continuing the operation in this way, we can get a sequence of minimal closed walks. And we call the members of this sequence the factors of the closed walk w .

For example, let w be a closed walk $\{u, a, b, c, d, c, d, u\}$, the first closed walk of w is $\{c, d, c\}$. And by deleting it, we get a new closed walk $w' = \{u, a, b, c, d, u\}$ which is a minimal closed walk. Then $\{c, d, c\}$ and $\{u, a, b, c, d, u\}$ are two factors of w .

When the closed walk w is in a tree, we find that all the factors of w have length two. A closed walk is called tree-like if all its factors have length two. In [10], Godsil gave a combinatorial interpretation about the moments of matching roots.

Lemma 3.1 ([10]). Let $m(G, x)$ be the matching polynomial of a graph G . Then the rational function $xm'(G, x)/m(G, x)$ is the generating function, in the variable x^{-1} , for the closed tree-like walks in G .

By [Lemma 3.1](#), we can get the following equation.

$$\frac{xm'(G, x)}{m(G, x)} = \sum_{i=1}^n \frac{x}{x - x_i} = \sum_{k=0}^{\infty} \left(\sum_{i=1}^n x_i^k \right) x^{-k}. \quad (3.1)$$

Then $\sum_{i=1}^n x_i^k$ is the number of closed tree-like walks with length k in G .

3.2. Convergence of the moments in expectation

We recall a weaker notion of convergence, named convergence in expectation, which is defined as follows. We say that EMDs converge in expectation to the semicircle law, if

$$\mathbf{E} \int_{\mathbb{R}} f(x) dF_n(x) \rightarrow \int_{\mathbb{R}} f(x) dF_{sc}(x) \quad (3.2)$$

as $n \rightarrow \infty$, for all bounded continuous function $f(x)$.

Theorem 3.2. For any positive integer k , $\lim_{n \rightarrow \infty} \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) = \int_{\mathbb{R}} x^k dF_{sc}(x)$.

Proof. For any positive integer k , the expected k th moment of the EMD is

$$\mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) = \mathbf{E} \frac{1}{n} \sum_{i=1}^n \lambda_i^k, \quad (3.3)$$

and the k th moment of the standard semicircle distribution is

$$\int_{-2}^2 x^k dF_{sc}(x).$$

Next, we just need to prove that for every fixed integer k ,

$$\mathbf{E} \frac{1}{n} \sum_{i=1}^n \lambda_i^k \rightarrow \int_{-2}^2 x^k dF_{sc}(x), \quad \text{as } n \rightarrow \infty.$$

And by Lemma 3.1, we have that

$$\mathbf{E} \frac{1}{n} \sum_{i=1}^n \lambda_i^k = \frac{1}{n^{1+k/2} p^{k/2}} \sum_w \mathbf{E}(X(w)),$$

where w denotes a tree-like closed walk with length k in the random graph $G_{n,p}$, and $X(w)$ is a random variable taking value 1 if w occurs and 0 otherwise.

When k is an odd number, then $\int_{-2}^2 x^k dF_{sc}(x) = 0$. Since w is a tree-like closed walk, the length of w must be even and for any edge of $G_{n,p}$, the total number of times that this edge appears in w is even. Then $\frac{1}{n^{1+k/2} p^{k/2}} \sum_w \mathbf{E}(X(w)) = 0$.

Now considering the even number $k = 2m$ where $m \geq 1$. We have

$$\begin{aligned} \int_{-2}^2 x^k dF_{sc}(x) &= \frac{1}{2\pi} \int_{-2}^2 x^k \sqrt{4-x^2} dx = \frac{1}{\pi} \int_0^2 x^{2m} \sqrt{4-x^2} dx \\ &= \frac{2^{2m+1}}{\pi} \cdot \frac{\Gamma(m+1/2)\Gamma(3/2)}{\Gamma(m+2)} = \frac{1}{m+1} \binom{2m}{m}, \end{aligned}$$

where $\Gamma(x)$ is the standard Gamma function. Let t denote the number of distinct vertices in a tree-like walk w . It is easy to see that t is no more than $m+1$. We divide the discussion into two cases.

Case 1. For a tree-like walk w with $t \leq m$, it is easy to check that when the underlying set of vertices is fixed, the number of this kind of closed walks is less than t^k and the edges of w induce a connected graph H_w with order t in $G_{n,p}$. Then we have (when n is sufficiently large such that $np \geq 1$)

$$\begin{aligned} \frac{1}{n^{1+k/2} p^{k/2}} \sum_{t=1}^m \sum_{|w=\{i_1, \dots, i_k\}|=t} \mathbf{E}(X(w)) &\leq \frac{1}{n^{1+m} p^m} \sum_{t=1}^m n^t \cdot t^k \cdot \mathbf{E}(X(H_w)) \\ &\leq \frac{1}{n^{1+m} p^m} \sum_{t=1}^m n^t \cdot t^k \cdot p^{t-1} \\ &\leq \frac{1}{n^{1+m} p^m} \cdot m \cdot n^m \cdot m^{2m} \cdot p^{m-1} \\ &= \frac{m^{2m+1}}{np} = o\left(\frac{1}{np}\right). \end{aligned}$$

Case 2. For a tree-like walk w with $t = m+1$, we know that each edge of the closed walk w appears twice, and the number of distinct edges in the closed walk w is m . The number of such kind of closed walks can be determined by the following lemma.

Lemma 3.3 ([2]). The number of the closed walks of length $2m$ on $m + 1$ vertices which satisfy that each edge and its inverse edge both appear once in the closed walks is $\frac{n(m+1)}{m+1} \binom{2m}{m}$, where $n_{(m+1)} = n(n-1) \cdots (n-m)$.

Then, we obtain

$$\begin{aligned} \frac{1}{n^{1+k/2} p^{k/2}} \sum_{|w=\{i_1, \dots, i_k\}|=m+1} \mathbf{E}(X(w)) &= \frac{1}{n^{1+k/2} p^{k/2}} \frac{n(n-1) \cdots (n-m)}{m+1} \binom{2m}{m} \\ &= \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m}{n}\right) \frac{1}{m+1} \binom{2m}{m}. \end{aligned}$$

When n is large enough, we have

$$\frac{1}{n^{1+k/2} p^{k/2}} \sum_{|w=\{i_1, \dots, i_k\}|=m+1} \mathbf{E}(X(w)) = (1 + o(1)) \frac{1}{m+1} \binom{2m}{m}.$$

By the above analysis, we have

$$\frac{1}{n^{1+k/2} p^{k/2}} \sum_w \mathbf{E}(X(w)) = \begin{cases} 0 & \text{if } k = 2m + 1; \\ \frac{1}{m+1} \binom{2m}{m} (1 + o(1)) + O\left(\frac{1}{np}\right) & \text{if } k = 2m. \end{cases}$$

Since $np \rightarrow \infty$ as $n \rightarrow \infty$, it follows that when $n \rightarrow \infty$,

$$\mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) \longrightarrow \int_{\mathbb{R}} x^k dF_{sc}(x).$$

The proof is thus complete. ■

3.3. Almost sure convergence of the moments

In this section, we prove the following theorem.

Theorem 3.4. For any positive integer k , $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} x^k dF_n(x) = \int_{\mathbb{R}} x^k dF_{sc}(x)$ a.s.

Proof. To prove that the moments of EMDs converge almost surely to the moments of the semicircle distribution, by Borel–Cantelli Lemma ([3], P. 60), it will be sufficient to show that

$$\sum_{n=1}^{\infty} P \left(\left| \int_{\mathbb{R}} x^k dF_n(x) - \int_{\mathbb{R}} x^k dF_{sc}(x) \right| > \epsilon \right) < \infty. \quad (3.4)$$

By the triangle inequality, we have

$$\begin{aligned} &P \left(\left| \int_{\mathbb{R}} x^k dF_n(x) - \int_{\mathbb{R}} x^k dF_{sc}(x) \right| > \epsilon \right) \\ &\leq P \left(\left| \int_{\mathbb{R}} x^k dF_n(x) - \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) \right| + \left| \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) - \int_{\mathbb{R}} x^k dF_{sc}(x) \right| > \epsilon \right) \\ &\leq P \left(\left| \int_{\mathbb{R}} x^k dF_n(x) - \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) \right| > \epsilon/2 \right) + P \left(\left| \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) - \int_{\mathbb{R}} x^k dF_{sc}(x) \right| > \epsilon/2 \right). \end{aligned}$$

By Theorem 3.2, we have

$$\sum_{n=1}^{\infty} P \left(\left| \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) - \int_{\mathbb{R}} x^k dF_{sc}(x) \right| > \epsilon/2 \right) < \infty.$$

Each term in the above sum is either 0 or 1 and only finitely many terms are ones. Otherwise, there is a subsequence of $\{\mathbf{E} \int_{\mathbb{R}} x^k dF_n(x)\}$ which does not converge to $\int_{\mathbb{R}} x^k dF_{sc}(x)$. By Chebyshev's inequality, we have

$$P \left(\left| \int_{\mathbb{R}} x^k dF_n(x) - \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) \right| > \epsilon/2 \right) < \frac{4}{\epsilon^2} \mathbf{Var} \int_{\mathbb{R}} x^k dF_n(x).$$

Next, we consider the variance of the moments of EMDs. Let $E(w)$ be the edge set of w and $V(w)$ the vertex set of w . Then

$$\begin{aligned}\text{Var} \int_{\mathbb{R}} x^k dF_n(x) &= \mathbf{E} \left(\int_{\mathbb{R}} x^k dF_n(x) \right)^2 - \left(\mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) \right)^2 \\ &= \frac{1}{n^{2+k} p^k} \sum_{w_1, w_2} (\mathbf{E}(X(w_1, w_2)) - \mathbf{E}(X(w_1))\mathbf{E}(X(w_2))),\end{aligned}$$

where w_1, w_2 are both tree-like closed walks with length k , and $X(w_1, w_2)$ is a random variable taking value 1 if w_1, w_2 both occur and 0 otherwise. If the tree-like closed walks w_1 and w_2 are edge-disjoint, then $\mathbf{E}(X(w_1, w_2)) = \mathbf{E}(X(w_1))\mathbf{E}(X(w_2))$. Hence, we only need to consider the pairs of tree-like closed walks which share at least one edge. Then $|V(w_1) \cup V(w_2)| \leq |V(w_1)| + |V(w_2)| - 2$. Since $|V(w)| \leq k/2 + 1$, the number of pairs of tree-like closed walks which contribute to the sum is no more than $O(n^k)$. It follows that

$$\frac{1}{n^{2+k} p^k} \sum_{w_1, w_2} (\mathbf{E}(X(w_1, w_2)) - \mathbf{E}(X(w_1))\mathbf{E}(X(w_2))) = \frac{1}{n^{2+k}} O(n^k) O(1) = O(n^{-2})$$

and

$$P \left(\left| \int_{\mathbb{R}} x^k dF_n(x) - \mathbf{E} \int_{\mathbb{R}} x^k dF_n(x) \right| > \epsilon/2 \right) = \frac{4}{\epsilon^2} O(n^{-2}).$$

Then for each $\epsilon > 0$, we have $\sum_{n=1}^{\infty} \frac{4}{\epsilon^2} O(n^{-2}) < \infty$.

Thus, Eq. (3.4) follows and the proof is complete. ■

4. The proof of Theorem 2.3

In this section, we give a proof of Theorem 2.3. Before proceeding, we recall a useful lemma.

Lemma 4.1 ([3], P. 198). *If the distribution function F_n converges weakly to the distribution function F , and F is everywhere continuous, then $F_n(x)$ converges to $F(x)$ uniformly in x .*

Proof of Theorem 2.3. For positive constants M and δ , we define a bounded continuous function

$$f(x) = \begin{cases} |x| & \text{if } -M \leq x \leq M; \\ M - \frac{M}{\delta}(x - M) & \text{if } M < x \leq M + \delta; \\ M + \frac{M}{\delta}(x + M) & \text{if } -M - \delta \leq x < -M; \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

Then by Theorem 2.2, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) dF_n(x) = \int_{\mathbb{R}} f(x) dF_{sc}(x) \quad a.s. \quad (4.2)$$

Let $M > 4$ and $\delta > 0$. And it follows that $\int_{|x| \geq M} f(x) dF_{sc}(x) = 0$. Now we consider the following inequalities.

$$\left| \int_{|x| \leq M} |x| dF_n(x) - \int_{|x| \leq M} |x| dF_{sc}(x) \right| \leq \left| \int_{\mathbb{R}} f(x) dF_n(x) - \int_{\mathbb{R}} f(x) dF_{sc}(x) \right| + 2 \left| \int_{M < x \leq M + \delta} f(x) dF_n(x) \right|$$

where

$$\begin{aligned}2 \left| \int_{M < x \leq M + \delta} f(x) dF_n(x) \right| &\leq 2M \left| \int_{M < x \leq M + \delta} dF_n(x) \right| \\ &\leq 2M \left| F_n(M + \delta) - F_{sc}(M + \delta) + F_{sc}(M) - F_n(M) + \int_{M < x \leq M + \delta} dF_{sc}(x) \right| \\ &\leq 2M \left(\left| F_n(M + \delta) - F_{sc}(M + \delta) \right| + \left| F_{sc}(M) - F_n(M) \right| + \left| \int_{M < x \leq M + \delta} dF_{sc}(x) \right| \right).\end{aligned}$$

From Eq. (2.1), we know that for any small number $\epsilon > 0$, there exists a number N_1 such that $\left| \int_{\mathbb{R}} f(x) dF_n(x) - \int_{\mathbb{R}} f(x) dF_{sc}(x) \right| < \frac{\epsilon}{3}$ for all $n > N_1$. Since $F_{sc}(x)$ is everywhere continuous, by Lemma 4.1, there exist a number $N_2 > 0$ and a number

$\delta_1 > 0$ such that, when $n > N_2$ and $0 < \delta < \delta_1$, $|F_n(x) - F_{sc}(x)| < \frac{\varepsilon}{12M}$ for all x , and $|\int_{M < x \leq M+\delta} dF_{sc}(x)| < \frac{\varepsilon}{12M}$. Hence, when $n > \max\{N_1, N_2\}$ and $0 < \delta < \delta_1$, we have

$$\left| \int_{|x| \leq M} |x| dF_n(x) - \int_{|x| \leq M} |x| dF_{sc}(x) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, we have

$$\lim_{n \rightarrow \infty} \int_{|x| \leq M} f(x) dF_n(x) = \int_{|x| \leq M} f(x) dF_{sc}(x) \quad a.s. \quad (4.3)$$

By the similar method, we can prove that

$$\lim_{n \rightarrow \infty} \int_{|x| \leq M} x^2 dF_n(x) = \int_{|x| \leq M} x^2 dF_{sc}(x) \quad a.s.$$

Since

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} x^2 dF_n(x) = \int_{\mathbb{R}} x^2 dF_{sc}(x) \quad a.s.$$

We obtain

$$\lim_{n \rightarrow \infty} \int_{|x| > M} x^2 dF_n(x) = \int_{|x| > M} x^2 dF_{sc}(x) \quad a.s.$$

Moreover, we know that $\int_{|x| > M} x^2 dF_{sc}(x) = 0$ and $0 \leq \int_{|x| > M} |x| dF_n(x) \leq \int_{|x| > M} x^2 dF_n(x)$. It follows that

$$\lim_{n \rightarrow \infty} \int_{|x| > M} |x| dF_n(x) = 0 = \int_{|x| > M} |x| dF_{sc}(x) \quad a.s. \quad (4.4)$$

Combining Eqs. (4.3) and (4.4), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |x| dF_n(x) = \int_{\mathbb{R}} |x| dF_{sc}(x) \quad a.s. \quad (4.5)$$

Now, it is time to estimate the matching energy of random graphs.

$$\begin{aligned} \frac{ME(G_{n,p})}{n^{3/2}p^{1/2}} &= \frac{1}{n^{3/2}p^{1/2}} \sum_{i=1}^n |x_i| = \frac{1}{n} \sum_{i=1}^n |\lambda_i| \\ &= \int |x| dF_n(x) \xrightarrow{n \rightarrow \infty} \int |x| \rho_{sc}(x) dx \quad a.s. \\ &= \frac{1}{2\pi} \int_{-2}^2 |x| \sqrt{4-x^2} dx \\ &= \frac{8}{3\pi}. \end{aligned}$$

The proof is thus complete. ■

Appendix. Theorem 3.4 implies Theorem 2.2

Proof. Let f be a bounded continuous function on $\mathbb{R}$. Let $M > 4$ and $\delta > 0$. By the Weierstrass approximation theorem, there exists a polynomial P_δ such that $|f(x) - P_\delta(x)| < \delta$ for all $|x| \leq M$.

$$\left| \int_{\mathbb{R}} f dF_n(x) - \int_{\mathbb{R}} f dF_{sc}(x) \right| \leq \left| \int_{\mathbb{R}} P_\delta d(F_n(x) - F_{sc}(x)) \right| + 2\delta + \left| \int_{[-M,M]^c} (f - P_\delta) dF_n(x) \right|. \quad (A.1)$$

Suppose the degree of P_δ is a . Since f is bounded, we have $f(x) < Cx^a$, for $|x| \geq M$ and some constant C . Then we have

$$\left| \int_{[-M,M]^c} (f - P_\delta) dF_n(x) \right| \leq \left| \int_{[-M,M]^c} C'|x|^a dF_n(x) \right| \leq C' \frac{1}{M^a} \int_{\mathbb{R}} x^{2a} dF_n(x)$$

where C' is a constant.

As $n \rightarrow \infty$, we have that the first part of Eq. (A.1) converges to zero, and the last part

$$\lim_{n \rightarrow \infty} C' \frac{1}{M^a} \int x^{2a} dF_n(x) \leq C' \frac{1}{M^a} \int_{-2}^2 x^{2a} dF_{sc}(x) \leq 4C' \frac{4^a}{M^a}.$$

By Theorem 3.4, for every $\epsilon > 0$, there exists a number N_1 such that for $n \geq N_1$,

$$\left| \int P_\delta d(F_n(x) - F_{sc}(x)) \right| < \epsilon/3. \quad (\text{A.2})$$

There exists a number M_1 , such that for $M > M_1$,

$$4C' \frac{4^a}{M^a} < \epsilon/3. \quad (\text{A.3})$$

Let $\delta < \frac{\epsilon}{6}$. By Eqs. (A.1), (A.2) and (A.3), we have

$$\left| \int f dF_n(x) - \int f dF_{sc}(x) \right| < \epsilon.$$

The proof is thus complete. ■

References

- [1] G.W. Anderson, A. Guionnet, O. Zeitouni, *An Introduction to Random Matrices*, Cambridge University Press, 2010.
- [2] Z. Bai, J.W. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, Springer, New York, 2009.
- [3] P. Billingsley, *Probability and Measure*, John Wiley & Sons, 2008.
- [4] X. Chen, X. Li, H. Lian, The skew energy of random oriented graphs, *Linear Algebra Appl.* 438 (2013) 4547–4556.
- [5] L. Chen, Y. Shi, Maximal matching energy of tricyclic graphs, *MATCH Commun. Math. Comput. Chem.* 73 (2015) 105–119.
- [6] W. Du, X. Li, Y. Li, The Laplacian energy of random graphs, *J. Math. Anal. Appl.* 368 (2010) 311–319.
- [7] W. Du, X. Li, Y. Li, Various energies of random graphs, *MATCH Commun. Math. Comput. Chem.* 64 (2010) 251–260.
- [8] W. Du, X. Li, Y. Li, The energy of random graphs, *Linear Algebra Appl.* 435 (2011) 2334–2346.
- [9] E.J. Farrell, An introduction to matching polynomial, *J. Combin. Theory Ser. B* 27 (1979) 75–86.
- [10] C. Godsil, Matchings and walks in graphs, *J. Graph Theory* 5 (1981) 285–297.
- [11] C. Godsil, I. Gutman, On the theory of the matching polynomial, *J. Graph Theory* 5 (1981) 137–144.
- [12] C. Godsil, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.
- [13] I. Gutman, Acyclic systems with extremal Hückel π -electron energy, *Theor. Chim. Acta* 45 (1977) 79–87.
- [14] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sect. Forschungsz. Graz* 103 (1978) 1–22.
- [15] I. Gutman, The matching polynomial, *MATCH Commun. Math. Comput. Chem.* 6 (1979) 75–91.
- [16] I. Gutman, M. Milun, N. Trinajstić, Graph theory and molecular orbitals 19, non-parametric resonance energies of arbitrary conjugated systems, *J. Amer. Chem. Soc.* 99 (1977) 1692–1704.
- [17] I. Gutman, S. Wagner, The matching energy of a graph, *Discrete Appl. Math.* 160 (2012) 2177–2187.
- [18] O.J. Heilmann, E.H. Lieb, Theory of monomer–dimer systems, *Comm. Math. Phys.* 25 (1972) 190–232.
- [19] S. Ji, X. Li, Y. Shi, Extremal matching energy of bicyclic graphs, *MATCH Commun. Math. Comput. Chem.* 70 (2013) 697–706.
- [20] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.
- [21] S. Li, W. Yan, The matching energy of graphs with given parameters, *Discrete Appl. Math.* 162 (2014) 415–420.
- [22] H. Li, Y. Zhou, L. Su, Graphs with extremal matching energies and prescribed parameters, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 239–248.
- [23] V. Nikiforov, The energy of graphs and matrices, *J. Math. Anal. Appl.* 326 (2007) 1472–1475.
- [24] T. Tao, Topics in Random Matrix Theory, in: *Graduate Studies in Mathematics*, vol. 132, American Mathematical Society, 2012.
- [25] E.P. Wigner, On the distribution of the roots of certain symmetric matrices, *Ann. of Math.* 67 (1958) 325–327.