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Properly colored notions of connectivity - a dynamic survey

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Properly colored notions of connectivity - a dynamic survey

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Abstract

A path in an edge-colored graph is properly colored if no two consecutive edges receive the same color. In this survey, we gather results concerning notions of graph connectivity involving properly colored paths.

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If you have corrections, updates or new results which fit the scope of this work, please contact Colton Magnant at cmagnant@georgiasouthern.edu.

1 Introduction

An edge-colored graph is said to be *properly colored* if no two adjacent edges share a color. An edge-colored connected graph G is called *properly connected* if between every pair of distinct vertices, there exists a path that is properly colored. The *proper connection number* of a connected graph G, defined in [4] and also studied in [1] and [20], is the minimum number of colors needed to color the edges of G to make it properly connected.

When building a communication network between wireless signal towers, one fundamental requirement is that the network is connected. If there cannot be a direct connection between two towers A and B, say for example if there is a mountain in between, there must be a route through other towers to get from A to B. As a wireless transmission passes through a signal tower, to avoid interference, it would help if the incoming signal and the outgoing signal do not share the same frequency. Suppose we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and assign a color to each edge based on the assigned frequency used for the communication. Then the number of frequencies needed to assign the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph.

Aside from the above application, properly colored paths and cycles appear in a variety of other fields including genetics [10, 11, 12] and social sciences [8]. There is also a good survey [2] dealing with the case where two colors are used on the edges. More recently, there has also been another survey of the area in Chapter 16 of [3].

Unless otherwise stated, we focus on coloring edges so "coloring" will mean edge-coloring. In most cases, k will generally be used to denote the number of colors used on the edges and n will generally be used to denote the order of G, that is, the number of vertices. Also define the color degree $d^c(v)$ to be the number of colors on edges incident to v. The connectivity of a graph G, denoted by $\kappa(G)$, is the minimum order of a set of vertices such that its removal results in at least two components. The chromatic number $\chi(G)$ (and similarly edge chromatic number $\chi'(G)$) is the minimum number of colors needed to color the vertices (respectively edges) so that no two adjacent vertices (respectively edges) receive the same color. For all other standard terminology, we refer the reader to [6].

The notion of proper edge-colorings has been extremely popular since the classical work of Vizing [24]. More recently, several works have considered properly colored subgraphs as opposed to looking at the entire graph. See [2] for a survey of work concerning properly colored cycles and paths in graphs and multigraphs.

The definition and study of the proper connection number was inspired by the rainbow connection number, defined by Chartrand et al. in [5]. A path is called *rainbow* if no two edges in the path share a color. The *rainbow connection number* of a graph G, denoted by $rc(G)$, is the minimum number of colors needed to color the graph so that between each pair of vertices, there is a rainbow path. By replacing "rainbow" with "proper", it is easy to see where the definition of the proper connection number originated. Furthermore, the *strong rainbow connection number*, denoted by $src(G)$, is the minimum number of colors needed to color the graph so that between every pair of vertices, there is a rainbow colored geodesic (shortest path).

More generally, a graph G is said to be *k-properly connected* if between every pair of distinct vertices, there exist k internally disjoint properly colored paths. The *k-proper connection number* of a k-connected graph G, denoted by $pc_k(G)$, is the minimum number of colors needed to color the edges of G to make it k-properly connected. For ease of notation and since many results concern only the 1-proper connection number, we let $pc(G) = pc_1(G)$.

2 General Results

Clearly $pc(G) = 1$ if and only if $G = K_n$ and $pc(G) = n - 1$ if and only if $G = K_{1,n-1}$. It is also clear that $pc_k(G)$ is monotone non-increasing under the operation of edge addition. The graphs G with $pc(G) = n - 2$ were classified by Huang et al. Here we let C_n and S_n denote the cycle and star on n vertices respectively.

Theorem 2.1 ([17]) *Let G be a connected graph on n vertices. Then $pc(G) = n - 2$ if and only if G is one of the following graphs:*

- *the unique tree with one vertex of degree $n - 2$,*

- C_3,
- C_4,
- $C_4 + e$,
- $S_4 + e$, *or*
- $S_5 + e$.

There is an easy lower bound on $pc(G)$ using the maximum number of bridges (cut edges) incident to a single vertex.

Theorem 2.2 ([1]) *Let G be a nontrivial connected graph that contains bridges. If b is the maximum number of bridges incident with a single vertex in G, then $pc(G) \geq b$.*

Trees also provide helpful bounds as seen in the following sequence of results.

Proposition 2.1 ([1]) *If T is a nontrivial tree, then $pc(T) = \chi'(T) = \Delta(T)$.*

Proposition 2.2 ([1]) *For a nontrivial connected graph G, $pc(G) \leq \min\{\Delta(T) :$ where T is a spanning tree of $G\}$.*

Corollary 2.3 ([1]) *If G contains a spanning path, then $pc(G) = 2$.*

In comparing the proper connection number to the rainbow connection number or the chromatic index, the following fact is clear from the definitions.

Fact 2.1 *For any connected graph G, $pc(G) \leq \chi'(G)$ and $pc(G) \leq rc(G)$.*

On the other hand, the following results show that there can be an arbitrarily large difference between these values.

Proposition 2.3 ([1]) *For any pair of positive integers a, b with $2 \leq a \leq b$, there is a connected graph G with $pc(G) = a$ and $rc(G) = b$.*

Proposition 2.4 ([1]) *For any pair of positive integers a, b with $2 \leq a \leq b$, there is a connected graph G with $pc(G) = a$ and $\chi'(G) = b$.*

Regarding complete multipartite graphs, the following result shows that the proper connection number is almost always 2.

Theorem 2.4 ([1]) *If G is a complete multipartite graph that is not a complete graph or a tree, then $pc(G) = 2$.*

The following results classify the graphs with proper connection number very large, that is, close to the number of edges in the graph. Here $S_{i,j}$ is the double star, that is the tree with exactly two internal vertices of degrees i and j respectively.

Theorem 2.5 ([20]) *Let G be a connected graph of size $m \geq 3$. Then $pc(G) = m - 1$ if and only if G is the double star $S_{2,m-1}$.*

Theorem 2.6 ([20]) *Let G be a connected graph of size $m \geq 4$. Then $pc(G) = m - 2$ if and only if G is the tree with max degree $m - 2$ or $G \in \{C_4, K_{1,3} + e, K_{1,4} + e\}$.*

Theorem 2.7 ([20]) *Let G be a connected graph of size $m \geq 5$. Then $pc(G) = m - 3$ if and only if G is one of the following graphs:*

- *a tree with $\Delta(G) = m - 3$,*
- *$G = K_{1,m-1} + e$ where $m \geq 6$,*
- *a star on $m - 3$ edges with the addition of a triangle sharing a vertex with one leaf of the star, or*
- *one of six specific graphs on at most 6 edges.*

3 Connectivity

In [4], several results were proven concerning the proper connection number using connectivity to provide density.

Theorem 3.1 ([4]) *If $\kappa(G) \geq 3$, then $pc(G) = 2$. If $\kappa(G) \geq 2$, then $pc(G) \leq 3$.*

This result was actually proven in a stronger form, to include the following "strong property", providing a special type of proper circuit containing any pair of vertices.

Definition 3.1 (Strong Property) *A proper connected graph is said to have the* strong property *if for every pair of vertices u and v, there exist two different (not necessarily disjoint) proper paths, say P and Q, from u to v such that the first colors used on each path are different and the last colors used on each path are different.*

The sharpness of Theorem 3.1 comes from the graph in Figure 1.

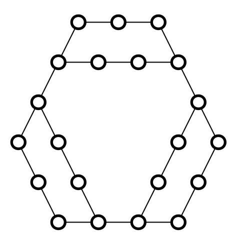

Figure 1: A 2-connected graph with $pc(G) = 3$

Theorem 3.1 was recently reproven in [16] using induction on the number of blocks in a connected bridgeless graph. More specifically, the following results were shown.

Theorem 3.2 ([16]) *If G is a connected bridgeless graph, then $pc(G) \leq 3$. Furthermore such a coloring can be produced with the strong property.*

In the case of bipartite graphs, it is easier to force the proper connection number down to 2.

Theorem 3.3 ([4]) *If G is a bipartite graph with $\kappa(G) \geq 2$, then $pc(G) = 2$. Furthermore, such a coloring can be produced with the strong property.*

This result was improved slightly in [16].

Theorem 3.4 ([16]) *If G is a connected bipartite bridgeless graph, then $pc(G) = 2$. Furthermore such a coloring can be produced with the strong property.*

As opposed to just connectivity, there have also been results using diameter to provide density in the graph.

Theorem 3.5 ([4]) *If $diam(G) = 2$ and G is 2-connected, then $pc(G) = 2$.*

We also believe the following to be true.

Conjecture 3.1 *If $diam(G) = 3$ and G is 2-connected, then $pc(G) = 2$.*

In the case where there are two connected spanning subgraphs that share very few edges, the following bound is known.

Theorem 3.6 ([15]) *If G is a graph containing two spanning subgraphs G_1 and G_2 such that $|E(G_1) \cap E(G_2)| = t$, then $pc(G) \leq t + 4$.*

4 Complements

Looking at the complement of the graph G, Huang et al. showed the following results.

Theorem 4.1 ([17]) *If G is a graph with $diam(G) \geq 4$, then $pc(\overline{G}) = 2$.*

Theorem 4.2 ([17]) *For a connected noncomplete graph G, if $\overline{G}$ does not belong to the following two cases: (i) $diam(\overline{G}) = 2, 3$, (ii) $\overline{G}$ contains exactly two connected components and one of them is trivial, then $pc(G) = 2$.*

For the next result, let G be a graph containing a vertex x and define $ecc(x)$ to be the eccentricity of x, that is, the maximum distance from another vertex to x. Also let $N_i(x)$ denote the set of vertices at distance i from x in G.

Theorem 4.3 ([17]) *Let G be a connected graph with $diam(G) = 3$ and let x be a vertex of G such that $ecc(x) = 3$. We have $pc(\overline{G}) = 2$ for the two cases (i) $G = P_4$ or (ii) $N_2(x) = 1$ and $N_3(x) \geq 2$. For the remaining cases, $pc(\overline{G})$ can be arbitrarily large. Furthermore, if G is triangle-free, then $pc(\overline{G}) = 2$.*

The previous result leads to the following ones.

Corollary 4.4 ([17]) *Let G be a triangle-free graph with $diam(G) = 2$. If $\overline{G}$ is connected, then $pc(\overline{G}) = 2$.*

Theorem 4.5 ([17]) *If G is connected and not complete and $\overline{G}$ is triangle-free, then $pc(G) = 2$.*

Based on these and other results, a Nordhaus-Gaddum type result for the proper connection number was obtained. For this first result, C_n is the cycle on n vertices and S_n is the star on n vertices.

Theorem 4.6 ([17]) *Let G be a graph on n vertices. Then $pc(G) = n - 2$ if and only if G is one of the following 6 graphs: unique tree with a vertex of degree $n - 2$, C_3, C_4, $C_4 + e$, $S_4 + e$, $S_5 + 3$.*

Theorem 4.7 ([17]) *If both G and $\overline{G}$ are connected, then for all $n \geq 5$, $pc(G) + pc(\overline{G}) \leq n$ and equality holds if and only if G or $\overline{G}$ is the unique tree with a vertex of degree $n - 2$.*

5 Minimum Degree

When density is provided by a minimum degree assumption, the following result is sharp in terms of the minimum degree assumption.

Theorem 5.1 ([4]) *If G is a connected (non-complete) graph with $n \geq 68$ vertices and $\delta(G) \geq n/4$, then $pc(G) = 2$.*

Although the bound on the minimum degree in Theorem 5.1 is best possible, there is nothing to suggest that the bound $n \geq 68$ is required. In the following conjecture, the bound on n would be the best possible since $pc(K_{1,3}) = 3$.

Conjecture 5.1 *If G is a connected (non-complete) graph with $n \geq 5$ vertices and $\delta(G) \geq n/4$, then $pc(G) = 2$.*

Using a weaker minimum degree assumption, the following more general result was obtained.

Theorem 5.2 ([14]) *For $t \geq 5$, if G is a connected graph of order $n \geq t^2$ and $\delta(G) \geq n/t$, then $pc(G) \leq t - 2$.*

For bipartite graphs, the natural variation of Theorem 5.2 holds.

Theorem 5.3 ([14]) *For $t \geq 4$, if G is a connected bipartite graph of order $n \geq 2t^2$ and $\delta(G) \geq n/2t$, then $pc(G) \leq t - 2$.*

It seems as though a combination of minimum degree and connectivity assumptions may also force the proper connection number to be 2.

Conjecture 5.2 ([4]) *If G is not complete, $\kappa(G) = 2$ and $\delta(G) \geq 3$, then $pc(G) = 2$.*

Using the number of edges in the graph to provide density, the following results hold.

Theorem 5.4 ([16]) *If G is a connected graph of order $n \geq 14$ with $\binom{n-3}{2} + 4 \leq |E(G)| \leq \binom{n}{2} - 1$, then $pc(G) = 2$.*

Theorem 5.5 ([16]) *If G is a connected graph of order $n \geq 15$ with $|E(G)| \geq \binom{n-4}{2} + 5$, then $pc(G) \leq 3$.*

With the assumption that $\kappa(G) \geq 2$, the proper connection number can be at most 3, so in order to produce results using more colors, we must assume that the graph is only 1-connected.

Theorem 5.6 ([14]) *For $s \geq 2$, any connected $K_{1,s}$-free graph G has $pc(G) \leq s - 1$.*

6 Domination

Relating the proper connection number with domination, the following results were proven in [21]. Here a subset D of the vertices is called *two-way two-step dominating* if every pendant vertex is included in D and every vertex at distance at least 2 from D has at least two neighbors at distance 1 from D.

Theorem 6.1 ([21]) *If D is a connected two-way two-step dominating set of a graph G, then $pc(G) \leq pc(G[D]) + 2$.*

This yields the following corollary.

Corollary 6.2 ([21]) *If G is a connected graph of order $n \geq 4$ and with minimum degree δ, then $pc(G) \leq \frac{3n}{\delta+1} - 1$.*

A result similar to Theorem 6.1 was shown for *two-way dominating* sets, namely, a dominating set containing all pendant vertices.

Theorem 6.3 ([21]) *If D is a connected two-way dominating set of a graph G, then $pc(G) \leq pc(G[D]) + 2$.*

Related to these results, the following was proven for a couple of special classes of graphs.

Theorem 6.4 ([21]) *If G is a connected interval or circular arc graph with $\delta(G) \geq 2$, then $pc(G) \leq 3$ and this bound is sharp.*

7 Combinations of Graphs

By a graph *join*, denoted by $G + H$, we mean adding all possible edges between a copy of G and a copy of H. The following was shown for joins.

Theorem 7.1 ([1]) *If G and H are nontrivial connected graphs such that $G + H$ is not complete, then $pc(G + H) = 2$.*

Concerning graph products, we define the *Cartesian product* of two graphs G and H, denoted by $G \Box H$, to be the graph with vertex set $V(G) \times V(H)$ and an edge from (u, v) to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Theorem 7.2 ([1]) *If G and H are nontrivial connected graphs, then $pc(G \Box H) = 2$.*

In [23], it seems that the authors got a more general result for the relationship between the graphs G and H and their Cartesian product: Let G and H be connected graphs each of order at least 2. Then

$$pc(G \Box H) \leq \min\{pc(G), pc(H)\} + 1,$$

and this bound is sharp. But in fact, they were not aware that actually this result is covered by Theorem 7.2, since for any nontrivial connected graphs G and H, $pc(G) \geq 1$ and $pc(H) \geq 1$.

Other similar results for the strong product, lexicographical product, the direct product as well as applications to grids, meshes, hypercubes and other classes of highly symmetric graphs were proven in [23]. Since these graph products contain the Cartesian product as a spanning subgraph, these results are also covered by Theorem 7.2.

8 Operations on Graphs

Given a graph G of order n with vertices $\{v_1, v_2, \ldots, v_n\}$, let α be a permutation of the set $[n]$. The permutation graph $P_\alpha(G)$ is the graph of order $2n$ obtained from two copies of G by joining a vertex v_i in the first copy to the corresponding vertex u_i and to $u_{\alpha(i)}$. Note that if α is the identity permutation, then $P_\alpha(G) = G \Box K_2$. The following results were shown for permutation graphs.

Theorem 8.1 ([1]) *If G is a nontrivial graph of order n containing a Hamiltonian path, then $pc(P_\alpha) = 2$ for any permutation α of $[n]$.*

Theorem 8.2 ([1]) *Every permutation graph of a star of order at least 4 has proper connection number 2.*

9 Random Graphs

For random graphs, the following results were shown in [15]. Here let $G(n,p)$ denote the Erdős-Renyi [13] random graph with n vertices and edges appearing with probability p. The first result follows easily from the fact that 3-connected.

Theorem 9.1 ([15]) *Almost all graphs have proper connection number* 2*.*

For the next result, it should be noted that if $\alpha(n)$ is sufficiently large, like $c \log \log n$, the result is almost immediate from the fact that the resulting graph will be Hamiltonian with high probability. Thus, the bulk of the work is done in the case where $\alpha(n)$ is tending to infinity very slowly.

Theorem 9.2 ([15]) *For sufficiently large n, if $p \geq \frac{\log n + \alpha(n)}{n}$ where $\alpha(n) \to \infty$, then $pc(G(n,p)) \leq 2$.*

10 Proper Distance

The *strong proper connection number* of a graph G, denoted by $spc(G)$ and defined in [1], is the minimum number of colors needed to color the edges of G so that between every pair of vertices, there is a properly colored geodesic.

First the following observations were made.

Proposition 10.1 ([1]) *Let G be a nontrivial connected graph of order n and size m. Then*

1. *$spc(G) = pc(G) = 1$ if and only if $G = K_n$;*
2. *$spc(G) = pc(G) = m$ if and only if $G = K_{1,m}$;*
3. *if G is a tree, then $spc(G) = pc(G) = \Delta(G)$;*
4. *if G is a connected graph with $diam(G) = 2$, then $spc(G) = src(G)$;*
5. *if b is the maximum number of bridges incident to a single vertex in G, then $spc(G) \geq b$.*

Clearly $pc(G) \leq spc(G)$ for any graph G. It turns out that the two values can also be arbitrarily far apart.

Theorem 10.1 ([1]) *For every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $pc(G) = a$ and $spc(G) = b$.*

A similar result holds for the strong proper connection number in relation to the chromatic index.

Theorem 10.2 ([1]) *For every triple of integers a, b and n with $2 \leq a \leq b < n$, there exists a connected graph of order n with $spc(G) = a$ and $\chi'(G) = b$.*

When the graph has large girth $g(G)$, it turns out that the bound above is sharp.

Proposition 10.2 ([1]) *If G is a connected graph with $g(G) \geq 5$, then $spc(G) = \chi'(G)$.*

Comparing $spc(G)$ to $src(G)$, we have the following results.

Theorem 10.3 ([1]) *For every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $spc(G) = a$ and $src(G) = b$.*

The authors of [20] also classified the graphs with strong proper connection number large, that is, close to the number of edges.

Theorem 10.4 ([20]) *Let G be a connected graph of size $m \geq 3$. Then $spc(G) = m - 1$ if and only if $G = S_{2,m-1}$.*

Theorem 10.5 ([20]) *Let G be a connected graph of size $m \geq 4$. Then $spc(G) = m - 2$ if and only if G is the tree with $\Delta(G) = m - 2$ or $G \in \{C_4, C_5, K_{1,m-1} + e\}$.*

Theorem 10.6 ([20]) *Let G be a connected graph of size $m \geq 5$. Then $spc(G) = m - 3$ if and only if G is one of the following graphs:*

- *a tree with $\Delta(G) = m - 3$,*
- *one of three classes of graphs looking like a star connected to a small cycle, or*
- *one of five graphs on at most 6 edges.*

Given a properly connected colored graph G, the *proper distance* between two vertices is the minimum length of a proper path between them and the *proper diameter* to be the maximum proper distance between two vertices.

Theorem 10.7 ([9]) *For a properly connected 2-coloring of the complete bipartite graph $K_{m,n}$, the proper diameter is either 2 or 4.*

The next results classify when the two values 2 and 4 are attainable for the proper diameter.

Theorem 10.8 ([9]) *For $m \geq n \geq 2$, there exists a properly connected 2-coloring of $K_{m,n}$ with proper diameter 2 if and only if $m < 2^n$.*

Theorem 10.9 ([9]) *If $\min\{m,n\} \geq 2$ and $\max\{m,n\} \geq 3$, then there exists a properly connected 2-coloring of $K_{m,n}$ with proper diameter 4.*

Other results for other classes of graphs like fans and ladders are also known.

11 Proper Vertex Connection

Notions of vertex proper connection, the vertex-coloring version of the proper connection number, have been defined and studied independently in [7] and [19]. A vertex-colored graph G is called *proper vertex k-connected* if every pair of vertices is connected by k internally disjoint paths, each of which has no two consecutive internal vertices of the same color. Define the *proper vertex k-connection number* of G, denoted by $pvc_k(G)$, to be the smallest number of (vertex) colors needed to make G proper vertex k-connected.

Some basic observations about $pvc_k(G)$ include the following.

Fact 11.1 ([19]) *For any graph G,*

$$0 \leq pvc_k(G) \leq \min\{\chi(G), rvc_k(G)\}$$

where $rvc(G)$ is the rainbow vertex connection number.

Proposition 11.1 ([19]) *If G is a nontrivial connected graph, then*

- *$pvc(G) = 0$ if and only if G is a complete graph, and*
- *$pvc(G) = 1$ if and only if $diam(G) = 2$.*

In particular, the following results were shown for cycles and wheels.

Theorem 11.1 ([19])

- *$pvc(C_3) = 0$, $pvc(C_4) = pvc(C_5) = 1$, and $pvc(C_n) = 2$ for $n \geq 6$.*
- *$pvc_2(C_3) = 1$, $pvc_2(C_n) = 2$ for even $n \geq 4$ and $pvc_2(C_n) = 3$ for odd $n \geq 5$.*
- *$pvc(W_3) = 0$ and $pvc(W_n) = 1$ for $n \geq 4$.*
- *$pvc_2(W_3) = 1$ and $pvc_2(W_n) = pvc(C_n)$ for all $n \geq 4$.*
- *$pvc_3(W_3) = 1$ and $pvc_3(W_n) = pvc_2(C_n)$ for all $n \geq 4$.*

The authors of [19] also have a result for pvc_k of complete multipartite graphs. More generally, the following results were shown for general graphs.

Theorem 11.2 ([19]) *If G is a nontrivial connected graph, then $pvc_1(G) = 2$ if and only if $diam(G) \geq 3$.*

Problem 11.1 ([19]) *For $k \geq 2$, is it true that $pc_k(G) \geq pvc_k(G)$ for any connected graph G?*

The work of [19] also considers the notion of *strong proper vertex connection* which requires that there exists a shortest path between any pair of selected vertices that is internally properly colored.

Theorem 11.3 ([19]) *If G is a nontrivial connected graph of order n, then $0 \leq spvc(G) \leq n - 2$ and equality holds on the right if and only if $G \in \{P_3, P_4\}$.*

The authors of [19] also classified the 12 graphs satisfying $spvc(G) = n - 3$. In relation to other related parameters, the following existence results are known.

Theorem 11.4 ([19]) *For every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $spvc(G) = a$ and $srvc(G) = b$.*

Theorem 11.5 ([19]) *For every triple of integers a, b and c with $2 \leq a \leq b \leq c$, there exists a connected graph G with $spvc(G) = a$, $\chi(G) = b$ and $\Delta(G) = c$.*

Similarly, the work of [7] considers a slightly different definition in which the entire paths (including end vertices) must be properly colored. Denote this variant $vpc(G)$. It turns out that this proper connection number can be classified by $S\chi_k(G)$, the minimum chromatic number of a spanning k-connected subgraph of G.

Theorem 11.6 ([7]) *Given a k-connected graph G, $vpc_k(G) = s\chi_k(G)$.*

Very recently, a notion of total proper connection was introduced in [18]. A graph is *total colored* if both edges and vertices receive colors. A path in a total colored graph is called a *total proper path* if no two adjacent edges of the path receive the same color, no two adjacent vertices of the path receive the same color and no vertex of the path receives the same color as an incident edge. A total colored graph G is called *total proper connected* if every two vertices in G are connected by a total proper path and the *total proper connection number* of a graph G, denoted by $tpc(G)$, is the minimum number of colors needed to total color G so that the coloring is total proper connected. Preliminary results on the subject include the following.

Proposition 11.2 ([18]) *If G is a nontrivial connected graph and H is a connected subgraph of G, then $tpc(G) \leq tpc(H)$.*

Proposition 11.3 ([18]) *Let G be a connected graph of order $n \geq 3$ containing at least one bridge. If b is the maximum number of bridges incident with a single vertex, then $tpc(G) \geq b + 1$.*

Theorem 11.7 ([18]) *If T is a tree of order $n \geq 3$, then $tpc(T) = \Delta(T) + 1$.*

These results immediately imply the following.

Corollary 11.8 ([18]) *For a nontrivial connected graph G, $tpc(G) \leq \min\{\Delta(T) + 1 :\ T$ is a spanning tree of $G\}$.*

More specifically, the following holds when G is traceable.

Corollary 11.9 ([18]) *If G is traceable, then $tpc(G) = 3$.*

For complete bipartite graphs, the following results were shown to hold.

Theorem 11.10 ([18]) *For $2 \leq m \leq n$, we have $tpc(K_{m,n}) = 3$.*

For total colorings, we extend the definition of the *strong property* (see Definition 3.1) to include the assumption that the first (and similarly last) edges of the paths must additionally not share a color with the starting (respectively ending) vertex. With this, we may state one of the main results of [18].

Theorem 11.11 ([18]) *Let G be a 2-connected graph. Then $tpc(G) \leq 4$ and there exists a total coloring of G with 4 colors with the strong property.*

For general (not necessarily 2-connected) graphs the authors of [18] obtained the following result.

Theorem 11.12 ([18]) *Let G be a connected graph and $\widetilde{\Delta}(G)$ be the maximum degree of a vertex that is an endpoint of a bridge of G. Then $tpc(G) \leq \widetilde{\Delta}(G) + 1$ if $\widetilde{\Delta}(G) \geq 4$ and $tpc(G) \leq 4$ otherwise.*

Regarding the minimum degree of G, the authors of [18] obtained the following result.

Theorem 11.13 ([18]) *If G is a connected graph of order n with minimum degree δ, then $tpc(G) \leq \frac{3n}{\delta+1} + 1$.*

Two interesting questions are also included. The first question was answered in the affirmative for the case where $k = 3$ in [18], which motivated the question itself.

Question 11.1 ([18]) *For $k \geq 4$, does there exist a graph G such that $tpc(G) = pc(G) = k$?*

An affirmative solution to the second question for the case when $t \leq 2$ was also noted in [18], motivating the question.

Question 11.2 ([18]) *For $t \geq 3$, does there exist a graph G such that $tpc(G) = pc(G) + t$?*

12 More Paths

Using a minimum degree assumption to provide density, the following was shown for $pc_2(G)$.

Theorem 12.1 ([16]) *If G is a connected graph with $n \geq 4$ vertices and $\delta(G) \geq n/2$, then $pc_2(G) = 2$.*

Not too surprisingly, the natural extension from the minimum degree to the degree sum also holds.

Theorem 12.2 ([16]) *If G is a connected graph with $n \geq 4$ vertices and $\sigma_2(G) \geq n$, then $pc_2(G) = 2$.*

With high connectivity in the underlying (uncolored) graph, it appears as though the k-proper connection number stays small.

Conjecture 12.1 ([4]) *If $k \geq 2$ and G is $2k$-connected, then $pc_k(G) \leq 3$.*

The case of Conjecture 12.1 where $k = 1$ follows from Theorem 3.1.

Proposition 12.1 ([4]) *If $k \geq 1$, $p = 2k - 1$ and $q > 2^p$, then $pc_k(K_{p,q}) > 2$.*

In light of this proposition, the following conjecture was made.

Conjecture 12.2 ([4]) *If G is a $2k$-connected bipartite graph with $k > 1$, then $pc_k(G) = 2$.*

This conjecture holds for complete bipartite graphs, as seen in the following result, but the general case remains open.

Theorem 12.3 ([4]) *If $G = K_{n,m}$, $m \geq n \geq 2k$ for $k \geq 1$, then $pc_k(G) = 2$.*

For the specific complete bipartite graphs $K_{3,n}$, the proper connection number for two paths is completely solved.

Theorem 12.4 ([4]) *If $G = K_{3,n}$, then*

$$pc_2(G) = \begin{cases} 2 & \text{if } 3 \leq n \leq 6, \\ 3 & \text{if } 7 \leq n \leq 8, \\ \lceil \sqrt[3]{n} \rceil & \text{if } n \geq 9. \end{cases}$$

For complete graphs, the following is known for any number of paths.

Theorem 12.5 ([4]) *If $G = K_n$ with $n \geq 4$, $k > 1$ and $n \geq 2k$, then $pc_k(G) = 2$.*

13 Directed Graphs

Much like the undirected version, a strongly connected directed graph is called *properly connected* if between every ordered pair of vertices, there is a directed properly colored path. Defined in [22], the *directed proper connection number* of a strongly connected directed graph G, denoted by $\overrightarrow{pc}(G)$, is the minimum number of colors needed to color the (directed) edges so that the graph is properly connected. Clearly $\overrightarrow{pc}(G) \geq 2$ for any G since a directed edge from u to v implies there is no directed edge from v to u so a directed path from v to u must use at least 2 colors.

It turns out that this number is always at most 3.

Theorem 13.1 ([22]) *If G is strongly connected, then $\overrightarrow{pc}(G) \leq 3$.*

In the case of a tournament, the number is always 2.

Theorem 13.2 ([22]) *If G is a strongly connected tournament, then $\overrightarrow{pc}(G) = 2$.*

The authors of [22] left the following problem open.

Problem 13.1 ([22]) *Classify the graphs with $\overrightarrow{pc}(G) = 2$.*

They also asked whether each of the graphs in the above classification contain a strongly connected bipartite subgraph which spans all but at most two vertices. As a special case of such a result, there is also the following conjecture.

Conjecture 13.1 ([22]) *If G is a strongly connected digraph with no even directed cycle, then $\overrightarrow{pc}(G) = 3$.*

With regards to color degree, the following was shown.

Theorem 13.3 ([22]) *If in a colored tournament G of odd order $n \geq 201$, each vertex has $\frac{n-1}{2}$ different colors on in-edges and $\frac{n-1}{2}$ different colors on out-edges, then G is properly connected.*

The bound on the number of colors on the in or out edges is actually best possible while the bound on n is likely not necessary.

Conjecture 13.2 (Implied in [22]) *If in an edge-colored tournament G of odd order n, each vertex has $\frac{n-1}{2}$ different colors on in-edges and $\frac{n-1}{2}$ different colors on out-edges, then G is properly connected.*

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