Nordhaus-Gaddum-type theorem for rainbow connection number of graphs*

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Abstract

An edge-colored graph G is *rainbow connected* if every two vertices of G are connected by a path whose edges have distinct colors. The *rainbow connection number* of G, denoted by $rc(G)$, is the minimum number of colors that are needed to make G rainbow connected. In this paper we give a Nordhaus-Gaddum-type result for the rainbow connection number. We prove that if G and $\overline{G}$ are both connected, then $4 \leq rc(G) + rc(\overline{G}) \leq n+2$. Examples are given to show that the upper bound is sharp for $n \geq 4$, and the lower bound is sharp for $n \geq 8$. Sharp lower bounds are also given for $n = 4, 5, 6, 7$, respectively.

Keywords: edge-colored graph, rainbow connection number, Nordhaus-Gaddum-type.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Undefined notation and terminology can be found in [1]. Let G be a nontrivial connected graph with an edge coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *rainbow path* if no two edges of the path are colored the same. The graph G is *rainbow connected* if for every two vertices u and v of G, there is a rainbow path connecting u and v. The minimum number of colors for which there is an edge coloring

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of G such that G is rainbow connected is called the *rainbow connection number*, denoted by $rc(G)$. Clearly, if a graph is rainbow connected, then it is also connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, just by assigning each edge a distinct color. Observe that if a connected graph G has n vertices, then $rc(G) \leq n-1$ since one may color the edges of a spanning tree of G with distinct colors and color the remaining edges with one of the colors already used. Moreover, we have $rc(G) \geq \mathrm{diam}(G)$, where $\mathrm{diam}(G)$ denotes the diameter of G. It is easy to see that $rc(G) = 1$ if and only if G is a clique, and $rc(G) = n-1$ if and only if G is a tree. It is also easy to see that a cycle with $k > 3$ vertices has rainbow connection number $\lceil k/2 \rceil$. Notice that if H is a connected spanning subgraph of G, then $rc(G) \leq rc(H)$, and this simple fact is used frequently. For more results on rainbow connections we refer to a new book [7].

A Nordhaus–Gaddum-type result is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name "Nordhaus–Gaddum-type" is given because Nordhaus and Gaddum [4] first established the following type of inequalities for chromatic number of graphs in 1956. They proved that if G and $\overline{G}$ are complementary graphs on n vertices whose chromatic numbers are $\chi(G)$ and $\chi(\overline{G})$, respectively, then

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1.$$

Since then, many analogous inequalities of other graph parameters have been considered, such as diameter [5], domination number [6], Wiener index and some other chemical indices [8], and so on. In this paper, we are concerned with analogous inequalities involving the rainbow connection number of graphs. We prove that

$$4 \leq rc(G) + rc(\overline{G}) \leq n+2.$$

The rest of this paper is organized as follows. At first, we give the upper bound and show that it is sharp for $n \geq 4$. Then we give the lower bound and show that it is also sharp for $n \geq 8$. Finally, sharp lower bound is also given for $n = 4, 5, 6, 7$, respectively.

2 Upper bound on $rc(G) + rc(\overline{G})$

We know that if G is a connected graph with n vertices, then the number of the edges in G must be at least $n-1$. If both G and $\overline{G}$ are connected, then n is at least 4, since

$$2(n-1) \leq e(G) + e(\overline{G}) = e(K_n) = \frac{n(n-1)}{2}. \qquad (*)$$

In the rest of the paper, we always assume that all graphs have at least 4 vertices, and both G and $\overline{G}$ are connected.

Lemma 1 *$rc(G)+rc(\overline{G})\leq n+2$ for $n=4,5$, and the bound is sharp.*

Proof. Note that $rc(G)\leq n-1$ and equality holds if and only if G is a tree. It follows that

$$rc(G)+rc(\overline{G})\leq 2(n-1),$$

and equality holds if and only if both G and $\overline{G}$ are trees.

If both G and $\overline{G}$ are trees, then $(*)$ must hold with equality. That is, n has to be 4. In this case, G and $\overline{G}$ are both paths, and

$$rc(G)+rc(\overline{G})=2(n-1)=6=4+2.$$

Then, for $n>4$ we have

$$rc(G)+rc(\overline{G})\leq 2n-3.$$

Particularly, for $n=5$

$$rc(G)+rc(\overline{G})\leq 2n-3=7=5+2.$$

To see that the bound is sharp, let G be a tree obtained from S_4 by attaching a pendant edge to one of the vertices of degree one, where S_4 is a star with 4 vertices. Then $rc(G)=4$. We observe that $\text{diam}(\overline{G})=3$ and it can be colored by three colors to make it rainbow connected. Thus $rc(\overline{G})=3$. Therefore, we have

$$rc(G)+rc(\overline{G})=7=2n-3=5+2.$$

∎

Lemma 2 *Let G be a nontrivial connected graph of order n, and $rc(G)=k$. Let $c: E(G)\to\{1,2,\cdots,k\}$ be a rainbow connected k-coloring of G. Add a new vertex u to G and join u to q vertices of G. The resulting graph is denoted by G'. Then, if $q\geq n+1-k$ we have $rc(G')\leq k$.*

Proof. Let X be the set of vertices adjacent to u. Suppose $X=\{x_1,x_2,\cdots,x_q\}$ and $V\backslash X=\{y_1,y_2,\cdots,y_{n-q}\}$. Since we have assumed $q\geq n+1-k$, it follows that $n-q\leq k-1$.

Since G is rainbow connected under the coloring c, for any y_i with $i\in\{1,2,\cdots,n-q\}$, there is a rainbow x_1-y_i path $P_{x_1y_i}$. For each $P_{x_1y_i}$, we find the last vertex on the path that belongs to X and denote by P_i the subpath from this vertex to y_i. Then P_i is a rainbow path whose vertices are in Y except the first vertex.

Let G_{x_i} be the union of the paths in $P_1, P_2, \cdots, P_{n-q}$ whose origin vertex is x_i, $1 \leq i \leq q$. Note that G_{x_i} may not be a tree, since these paths may have common edges and vertices other than x_i. If there is no path with origin vertex x_i, let G_{x_i} be a trivial graph with a single vertex x_i. Now G_{x_i} is a subgraph of G, and $v(G_{x_i}) \leq n-q+1 \leq k$. Suppose $V(G_{x_i}) = \{x_i, y_{i_1}, y_{i_2}, \cdots, y_{i_l}\}$.

We now distinguish two cases according to the number of colors appearing in G_{x_i}. In each of the cases we will color the edge ux_i such that for any $j \in \{1, 2, \cdots, l\}$, there is a rainbow $u - y_{i_j}$ path.

Case 1. The number of colors appearing in G_{x_i} is k. Then $e(G_{x_i}) \geq k$.

Subcase 1.1. $e(G_{x_i}) = k \geq v(G_{x_i})$.

In this case, G_{x_i} contains a cycle, and none of the edges of G_{x_i} are colored the same, which implies that G_{x_i} is rainbow connected. Let e be an edge in the cycle. Delete e and color the edge ux_i with the color $c(e)$. Now we have, for any $j \in \{1, 2, \cdots, l\}$, that there is a rainbow $u - y_{i_j}$ path.

Subcase 1.2. $e(G_{x_i}) > k \geq v(G_{x_i})$.

In this case, G_{x_i} contains a cycle, and there are two edges e_1 and e_2 with $c(e_1) = c(e_2)$.

If one of the edges, say e_1, is contained in a cycle of G_{x_i}, then by deleting it, we obtain a spanning subgraph G'_{x_i} of G_{x_i} with the same number of colors, but $e(G'_{x_i}) = e(G_{x_i}) - 1$. If $e(G'_{x_i}) = k$, then by the similar operation in Subcase 1.1, we can obtain a coloring of ux_i such that for any $j \in \{1, 2, \cdots, l\}$, there is a rainbow $u - y_{i_j}$ path. If $e(G'_{x_i}) > k$, then we consider the graph G'_{x_i} as opposed to G_{x_i}.

If neither e_1 nor e_2 is in a cycle, then they must be cut edges of G_{x_i}. Contract one of them, say e_1, and denote the resulting graph by G''_{x_i}. The number of colors appearing in G''_{x_i} is still k, but $v(G''_{x_i}) = v(G_{x_i}) - 1, e(G''_{x_i}) = e(G_{x_i}) - 1$. If $e(G''_{x_i}) = k$, then by the similar operation in Subcase 1.1, we can obtain a coloring of ux_i such that there still exists a rainbow $u - y_{i_j}$ path in G_{x_i} for any $j \in \{1, 2, \cdots, l\}$. Indeed, by the construction of G_{x_i}, we have that for any y_{i_j} there is a rainbow $x_i - y_{i_j}$ path P in G_{x_i}. Consider the path $ux_iPy_{i_j}$. If no two edges are colored the same, then it is a rainbow $u - y_{i_j}$ path. Otherwise, there is an edge $e \in E(P)$ such that $c(ux_i) = c(e)$. By the coloring of the edge ux_i, we have that e is contained in a cycle of G_{x_i}, and no two edges of the cycle are colored the same. By Subcase 1.1, we can obtain a rainbow $x_i - y_{i_j}$ path P' which does not contain e. Thus $ux_iP'y_{i_j}$ is a rainbow $u - y_{i_j}$ path. If $e(G''_{x_1}) > k$, then we consider the graph G''_{x_i} other than G_{x_i}.

Case 2. The number of colors appearing in G_{x_i} is less than k. Then we color the edge ux_i with a color not appearing in G_{x_i}.

No matter which cases happen, we can always color the edge ux_i with one of the colors $\{1, 2, \cdots, k\}$ such that for any $j \in \{1, 2, \cdots, l\}$, there is a rainbow $u - y_{i_j}$ path.

Now we obtain a k-coloring of G' and in this coloring G' is rainbow connected, since for each y_i there is an x_j such that $y_i \in G_{x_j}$ and the path ux_jP_i is a rainbow path connecting u and y_i. Therefore, $rc(G') \leq k$. ∎

Theorem 1 *$rc(G) + rc(\overline{G}) \leq n + 2$ for $n \geq 4$, and this bound is best possible.*

Proof. By induction on n. By Lemma 1, the result is true for $n = 4$ and 5. We assume that $rc(G) + rc(\overline{G}) \leq n + 2$ holds for complementary graphs on n vertices. To the union of connected graphs G and $\overline{G}$, a complete graph on the n vertices, we adjoin a new vertex u. Let q be the number of vertices of G which are adjacent to u. Then the number of vertices of $\overline{G}$ which are adjacent to u is $n - q$. If G' and $\overline{G'}$ are the resulting graphs with order $n + 1$, then

$$rc(G') \leq rc(G) + 1, \quad rc(\overline{G'}) \leq rc(\overline{G}) + 1.$$

Indeed, given a rainbow $rc(G)$-coloring of G, if we assign a new color to the edges between u and G, then the resulting coloring makes G' rainbow connected. Thus, $rc(G') + rc(\overline{G'}) \leq rc(G) + rc(\overline{G}) + 2 \leq n + 4$, and $rc(G') + rc(\overline{G'}) \leq n + 3$ except when

$$rc(G') = rc(G) + 1, \quad rc(\overline{G'}) = rc(\overline{G}) + 1.$$

In this case, by Lemma 2, $q \leq n - rc(G), n - q \leq n - rc(\overline{G})$. Hence, $rc(G) + rc(\overline{G}) \leq n$, from which $rc(G') + rc(\overline{G'}) \leq n + 2 < n + 3$. This completes the induction.

To see that the bound can be attained, let G be a tree obtained by joining the centers of two stars S_p and S_q by an edge uv, where u and v are the centers of S_p and S_q, and $p+q = n$, $p \geq 1$, $q \geq 1$. Then $rc(G) = n-1$. To compute the rainbow connection number of the complement of G, we assume that $X = V(S_p \backslash u) \cup \{v\}$ and $Y = V(S_q \backslash v) \cup \{u\}$. Then G is a bipartite graph with bipartition (X, Y). Thus $\overline{G}[X]$ and $\overline{G}[Y]$ are complete graphs. We assign color 1 to the edges of $\overline{G}[X]$, 2 to the edges of $\overline{G}[Y]$ and 3 to the edges between X and Y. The resulting coloring makes $\overline{G}$ rainbow connected. Thus $rc(\overline{G}) \leq 3$. On the other hand, since $\mathrm{diam}(\overline{G}) = d(u, v) = 3$, it follows that $rc(\overline{G}) = 3$. Therefore, we have $rc(G) + rc(\overline{G}) = n - 1 + 3 = n + 2$. ∎

3 Lower bound on $rc(G) + rc(\overline{G})$

As we have noted, $rc(G) = 1$ if and only if G is a complete graph. In this case, $\overline{G}$ is not connected. Thus, if both G and $\overline{G}$ are connected, then we have $rc(G) \geq 2$ and $rc(\overline{G}) \geq 2$. That is, $rc(G) + rc(\overline{G}) \geq 4$.

Proposition 1 *Let G and $\overline{G}$ be complementary connected graphs with $rc(G) = rc(\overline{G}) = 2$. Then*
(1) $diam(G) = diam(\overline{G}) = 2$.
(2) $2 \leq \delta(G) \leq \Delta(G) \leq n-3$, $2 \leq \delta(\overline{G}) \leq \Delta(\overline{G}) \leq n-3$.
(3) A vertex u in $N^1(v)$ can not be adjacent to all vertices of $N^2(v)$, where $N^1(v)$, $N^2(v)$ is the first and second neighborhood of a vertex v, respectively.

Proof. Since $2 \leq diam(G) \leq rc(G) = 2$, (1) clearly holds.

For (2), first, $\Delta(G) \neq n-1$, otherwise $\overline{G}$ is disconnected.
Second, $\delta(G) \neq 1$. Indeed, if $\delta(G) = 1$, let v be a vertex of degree one, and u be the vertex adjacent to v. Then u must be adjacent to all the other vertices since $\text{diam}(G) = 2$. Hence $d(u) = n-1$, a contradiction. Similarly, $\delta(\overline{G}) \neq 1$. That is, $\delta(\overline{G}) \geq 2$. Therefore, $\Delta(G) \leq n-1-\delta(\overline{G}) \leq n-3$, and similarly for $\Delta(\overline{G})$.

For (3), if u is adjacent to all vertices of $N^2(v)$, then u is not adjacent to them in $\overline{G}$, the distance between u and $N^2(v)$ is at least 2 in $\overline{G}$, and v is adjacent to all vertices of $N^2(v)$ but not to the vertices in $N^1(v)$ of $\overline{G}$. Therefore, $d_{\overline{G}}(u,v) \geq 3$, which contradicts (1). ∎

Theorem 2 *For $4 \leq n \leq 7$, there are no graphs G and $\overline{G}$ on n vertices such that $rc(G) = rc(\overline{G}) = 2$.*

Proof. We consider $n = 4, 5, 6, 7$, respectively.

Case 1. n=4.

In this case, there is only one pair of complementary connected graphs, each of which is isomorphic to P_4 and therefore has a rainbow connection number 3.

Case 2. n=5.

If $rc(G) = rc(\overline{G}) = 2$, then by Proposition 1, $2 \leq \delta(G) \leq \Delta(G) \leq n-3 = 2$. It implies that $G \cong C_5$ and $\overline{G} \cong C_5$. Since $rc(C_5) = 3$, there are no graphs G and $\overline{G}$ on 5 vertices such that $rc(G) = rc(\overline{G}) = 2$.

Case 3. n=6.

By Proposition 1, $2 \leq \delta(G) \leq \Delta(G) \leq n-3 = 3$, and so the possible degree sequences are:

$$\text{(a)}\begin{cases} d_G = (2,2,2,2,2,2) \\ d_{\overline{G}} = (3,3,3,3,3,3). \end{cases}$$

(b)$\begin{cases} d_G = (3,3,2,2,2,2) \\ d_{\overline{G}} = (2,2,3,3,3,3). \end{cases}$

The graph with degree sequence in (a) is a cycle of length 6, whose rainbow connection number is 3. Hence, there is no graph with degree sequence in (a) whose rainbow connection is 2.

The graph with degree sequence in (b) satisfying Proposition 1 has to be the graph shown in Figure 1:

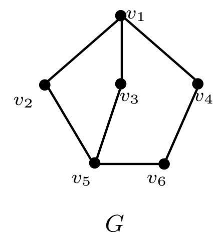

Figure 1: Graphs with degree sequence (3,3,2,2,2,2) satisfying Proposition 1.

The only 2-path between v_2 and v_4 is $v_2v_1v_4$, where and in what follows a 2-path means a path of length 2. Thus $c(v_1v_2) \neq c(v_1v_4)$. Similarly, $c(v_1v_3) \neq c(v_1v_4)$. It follows that $c(v_1v_2) = c(v_1v_3)$. If we consider the pairs of vertices (v_2, v_6) and (v_3, v_6), then we have $c(v_2v_5) = c(v_3v_5)$. However, now there is no rainbow $v_2 - v_3$ path. Therefore, $rc(G) \neq 2$.

Case 4. n=7.

By Proposition 1, $2 \leq \delta(G) \leq \Delta(G) \leq n-3 = 4$, and so the possible degree sequences are: (in the following argument, we use two colors to color the edges of the graphs)

(1)$\begin{cases} d_G = (4,4,4,4,4,4,4) \\ d_{\overline{G}} = (2,2,2,2,2,2,2). \end{cases}$

In this case, $\overline{G}$ is a cycle of length 7, $rc(\overline{G}) = 4$.

(2)$\begin{cases} d_G = (4,4,4,4,4,3,3) \\ d_{\overline{G}} = (2,2,2,2,2,3,3). \end{cases}$

The graphs with degree sequence (4,4,4,4,4,3,3) satisfying Proposition 1 are G_1 and G_2 shown in Figure 2. The distance between v_2 and v_5 in $\overline{G_1}$ and $\overline{G_2}$ is larger than 2. Thus $rc(\overline{G_1}) \neq 2$ and $rc(\overline{G_2}) \neq 2$.

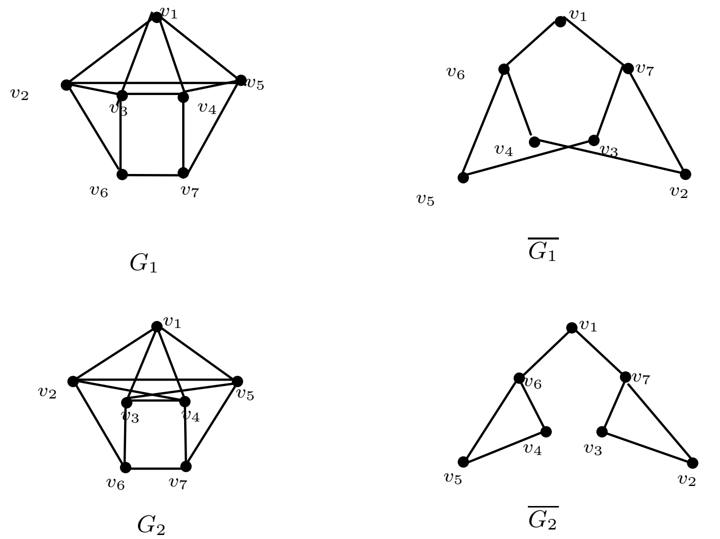

Figure 2: Graphs with degree sequence (4,4,4,4,4,3,3) satisfying Proposition 1.

$$(3)\begin{cases} d_G = (4,4,4,3,3,3,3) \\ d_{\overline{G}} = (2,2,2,3,3,3,3). \end{cases}$$

The graphs with degree sequence (4,4,4,3,3,3,3) satisfying Proposition 1 are subgraphs G_1' and G_2' of G_1 and G_2 shown in Figure 2 by deleting the edge v_2v_5. We observe that the distance between v_3 and v_4 in $\overline{G_1'}$ and $\overline{G_2'}$ is larger than 2. Thus $rc(\overline{G_1'}) \neq 2$ and $rc(\overline{G_2'}) \neq 2$.

$$(4)\begin{cases} d_G = (4,3,3,3,3,3,3) \\ d_{\overline{G}} = (2,3,3,3,3,3,3). \end{cases}$$

The graphs with degree sequence (4,3,3,3,3,3,3) satisfying Proposition 1 are G_1, G_2 and G_3 shown in Figure 3.

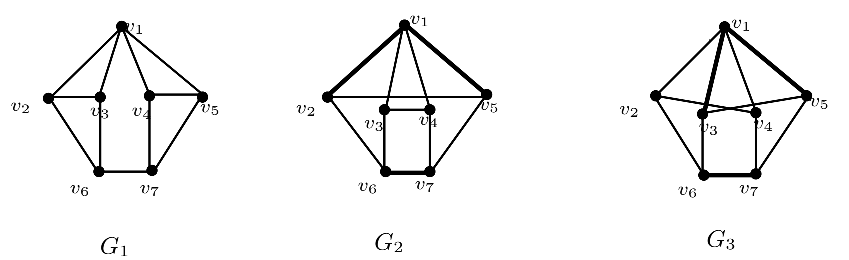

Figure 3: Graphs with degree sequence (4,3,3,3,3,3,3) satisfying Proposition 1.

Consider G_1. The only 2-path between v_3 and v_4 is $v_3v_1v_4$. Thus $c(v_1v_3) \neq c(v_1v_4)$. Similarly, $c(v_1v_3) \neq c(v_1v_5)$. It follows that $c(v_1v_4) = c(v_1v_5)$, say color 2. If we consider the pairs of vertices (v_2, v_4) and (v_3, v_4), then we have $c(v_1v_3) = c(v_1v_2) = 1$. If we consider the pairs of vertices $(v_2,v_7),(v_3,v_7)$, $(v_4,v_6),(v_5,v_6)$, then we have $c(v_2v_6) = c(v_3v_6) = c(v_4v_7) = c(v_5v_7)$. If $c(v_2v_6)=1$, then there is no rainbow v_1-v_6 path. If $c(v_2v_6)=2$, then there is no rainbow v_1-v_7 path. Therefore, $rc(G_1)\neq 2$.

Consider G_2. Color the heavy edges with color 2 and the other edges with color 1. It is easy to check that the coloring makes G_2 rainbow connected. Next consider the complement of G_2. The only 2-path between v_6 and v_7 is $v_6v_1v_7$. Thus $c(v_1v_6)\neq c(v_1v_7)$. Assume $c(v_1v_6)=1$ and $c(v_1v_7)=2$. If we consider the pairs of vertices (v_1,v_3), (v_1,v_2), (v_1,v_4), (v_1,v_5), then we have $c(v_3v_7)=c(v_2v_7)=1$ and $c(v_4v_6)=c(v_5v_6)=2$. If $c(v_2v_4)=1$, then there is no rainbow v_4-v_7 path. If $c(v_2v_4)=2$, then there is no rainbow v_2-v_6 path. Therefore, $rc(\overline{G_2})\neq 2$.

Consider G_3. Color the heavy edges with color 2 and the other edges with color 1. It is easy to check that the coloring makes G_3 rainbow connected. Next consider the complement of G_3. The only 2-path between v_6 and v_7 is $v_6v_1v_7$. Thus $c(v_1v_6)\neq c(v_1v_7)$. Assume $c(v_1v_6)=1$ and $c(v_1v_7)=2$. If we consider the pairs of vertices (v_1,v_3), (v_1,v_2), (v_1,v_4), (v_1,v_5), then we have $c(v_3v_7)=c(v_2v_7)=1$ and $c(v_4v_6)=c(v_5v_6)=2$. If $c(v_2v_5)=1$, then there is no rainbow v_5-v_7 path. If $c(v_2v_5)=2$, then there is no rainbow v_2-v_6 path. Therefore, $rc(\overline{G_3})\neq 2$.

$$(5)\begin{cases} d_G=(4,4,4,4,3,3,2) \\ d_{\overline{G}}=(2,2,2,2,3,3,4). \end{cases}$$

The graphs with degree sequence (4,4,4,4,3,3,2) satisfying Proposition 1 are G_1,G_2,G_3,G_4 shown in Figure 4.

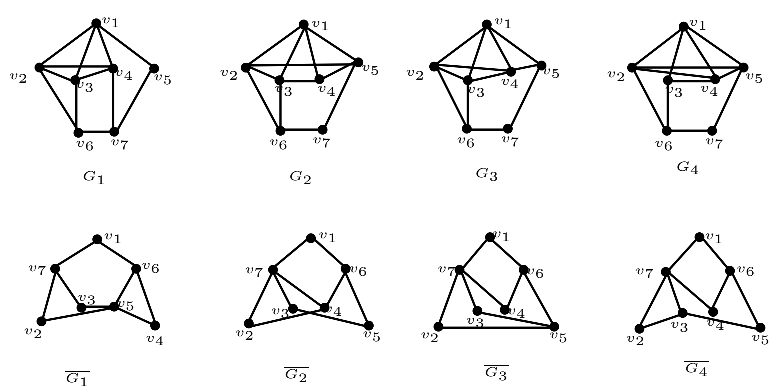

Figure 4: Graphs with degree sequence (4,4,4,4,3,3,2) satisfying Proposition 1.

Since $d_{\overline{G_1}}(v_4,v_7)=3$, $d_{\overline{G_2}}(v_2,v_5)=3$, $d_{\overline{G_4}}(v_2,v_6)=3$, it follows that $rc(\overline{G_1})\geq 3$, $rc(\overline{G_2})\geq 3$, $rc(\overline{G_4})\geq 3$.

Consider $\overline{G_3}$. The only 2-path between v_4 and v_5 is $v_4v_6v_5$. Thus $c(v_4v_6)\neq c(v_5v_6)$. Assume $c(v_4v_6)=1$ and $c(v_5v_6)=2$. If we consider the pairs of vertices (v_3,v_6) and

(v_2, v_6), then we have $c(v_3v_5) = c(v_2v_5) = 1$. If we consider the pairs of vertices (v_1, v_2), (v_1, v_3), (v_5, v_7), then we have $c(v_2v_7) = c(v_3v_7) = 2$. Moreover, if we consider the pairs of vertices (v_1, v_2) and (v_2, v_4), then we have $c(v_1v_7) = c(v_4v_7) = 1$. If $c(v_1v_6) = 1$, then there is no rainbow $v_1 - v_4$ path. If $c(v_1v_6) = 2$, then there is no rainbow $v_1 - v_5$ path. Therefore, $rc(\overline{G_3}) \neq 2$.

$$(6)\begin{cases} d_G = (4, 4, 3, 3, 3, 3, 2) \\ d_{\overline{G}} = (2, 2, 3, 3, 3, 3, 4). \end{cases}$$

The graphs with degree sequence (2,2,3,3,3,3,4) satisfying Proposition 1 are G_1, G_2, G_3 shown in Figure 5.

Consider G_1. The only 2-path between v_2 and v_7 is $v_2v_6v_7$. Thus $c(v_2v_6) \neq c(v_6v_7)$. Assume $c(v_2v_6) = 1$ and $c(v_6v_7) = 2$. If we consider the pair of vertices (v_3, v_7), then we have $c(v_3v_6) = 1$. If we consider the pairs of vertices (v_5, v_6), (v_4, v_7), (v_4, v_6), then we have $c(v_5v_7) = 1$, $c(v_4v_5) = 2$, $c(v_3v_4) = 2$. If we consider the pairs of vertices (v_2, v_5) and (v_3, v_5), then we have $c(v_1v_2) = c(v_1v_3)$. However, now there is no rainbow $v_2 - v_3$ path. Therefore, $rc(G_1) \neq 2$.

Consider G_2. The only 2-path between v_4 and v_6 is $v_4v_7v_6$. Thus $c(v_4v_7) \neq c(v_6v_7)$. Similarly, $c(v_5v_7) \neq c(v_6v_7)$. It follows that $c(v_4v_7) = c(v_5v_7)$. If we consider the pairs of vertices (v_2, v_4) and (v_2, v_5), then we have $c(v_1v_4) = c(v_1v_5)$. However, now there is no rainbow $v_4 - v_5$ path. Therefore, $rc(G_2) \neq 2$.

Consider G_3. Color the heavy edges with color 2 and the other edges with color 1. It is easy to check that the coloring makes G_3 rainbow connected. Next consider the complement of G_3. The only 2-path between v_6 and v_7 is $v_6v_1v_7$. Thus $c(v_1v_7) \neq c(v_1v_6)$. Assume $c(v_1v_7) = 1$ and $c(v_1v_6) = 2$. If we consider the pairs of vertices (v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_3, v_6), (v_4, v_7), then we have $c(v_2v_7) = c(v_3v_7) = 2$, $c(v_4v_6) = c(v_5v_6) = 1$, $c(v_3v_5) = 2$, $c(v_2v_4) = 1$. If $c(v_2v_5) = 1$, then there is no rainbow $v_2 - v_6$ path. If $c(v_2v_5) = 2$, then there is no rainbow $v_5 - v_7$ path. Therefore, $rc(\overline{G_3}) \neq 2$.

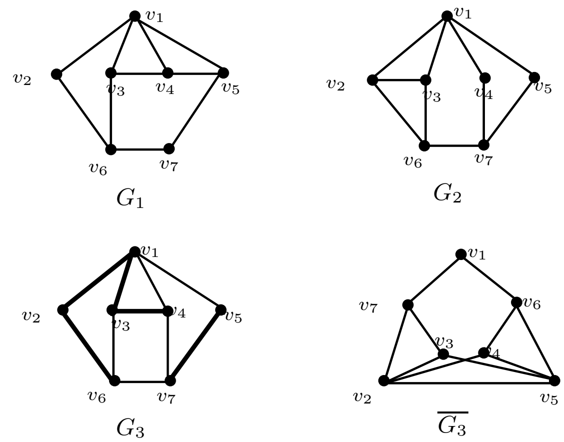

Figure 5: Graphs with degree sequence (2,2,3,3,3,3,4) satisfying Proposition 1.

$$(7)\begin{cases} d_G = (4,4,4,3,3,2,2) \\ d_{\overline{G}} = (2,2,2,3,3,4,4). \end{cases}$$

The graphs with degree sequence (2,2,2,3,3,4,4) satisfying Proposition 1 are G_1 and G_2 shown in Figure 6.

Consider G_1. The only 2-path between v_1 and v_7 is $v_1v_5v_7$. Thus $c(v_1v_5) \neq c(v_5v_7)$. Assume $c(v_1v_5) = 1$ and $c(v_5v_7) = 2$,. If we consider the pairs of vertices (v_5, v_6), (v_2, v_7), (v_3, v_7), (v_4, v_7), (v_2, v_5), (v_3, v_5), (v_4, v_5), then we have $c(v_6v_7) = 1,\ c(v_2v_6) = c(v_3v_6) = c(v_4v_6) = 2$, $c(v_1v_2) = c(v_1v_3) = c(v_1v_4) = 2$. However, now there is no rainbow $v_2 - v_3$ path. Therefore, $rc(G_1) \neq 2$.

Consider G_2. The only 2-path between v_2 and v_7 is $v_2v_6v_7$. Thus $c(v_2v_6) \neq c(v_6v_7)$. Similarly, $c(v_3v_6) \neq c(v_6v_7)$. It follows that $c(v_2v_6) = c(v_3v_6)$. If we consider the pairs of vertices (v_2, v_4) and (v_3, v_4), then we have $c(v_1v_2) = c(v_1v_3)$. However, now there is no rainbow $v_2 - v_3$ path. Therefore, $rc(G_2) \neq 2$.

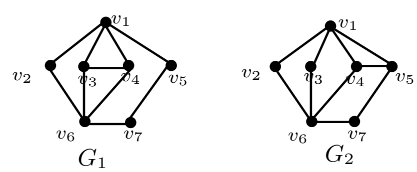

Figure 6: Graphs with degree sequence (2,2,2,3,3,4,4) satisfying Proposition 1.

$$(8)\begin{cases} d_G = (4,2,2,2,2,2,2) \\ d_{\overline{G}} = (2,4,4,4,4,4,4). \end{cases}$$

There is no graph with degree sequence (4,2,2,2,2,2,2) satisfying Proposition 1.

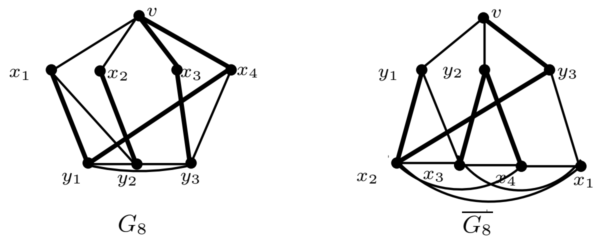

Figure 7: $rc(G) = rc(\overline{G}) = 2$ for $n = 8$.

$$(9)\begin{cases} d_G = (4,4,2,2,2,2,2) \\ d_{\overline{G}} = (2,2,4,4,4,4,4). \end{cases}$$

The graph G with degree sequence (4,4,2,2,2,2,2) satisfying Proposition 1 is the subgraph G' of G_1 depicted in Figure 6 by deleting the edge v_3v_4. If we consider the pairs of vertices (v_2, v_7) and (v_3, v_7), then we have $c(v_2v_6) = c(v_3v_6)$. If we consider the pairs of vertices (v_2, v_5) and (v_3, v_5), then we have $c(v_1v_2) = c(v_1v_3)$. However, now there is no rainbow $v_2 - v_3$ path. Therefore, $rc(G') \neq 2$.

$$(10)\begin{cases} d_G = (4,4,4,2,2,2,2) \\ d_{\overline{G}} = (2,2,2,4,4,4,4). \end{cases}$$

There is no graph with degree sequence (4,4,4,2,2,2,2) satisfying Proposition 1. ∎

Theorem 3 *For $n \geq 8$, the lower bound $rc(G) + rc(\overline{G}) \geq 4$ is best possible, that is, there are connected graphs G and $\overline{G}$ on n vertices such that $rc(G) = rc(\overline{G}) = 2$.*

Proof. For $n = 8$, see G_8 in Figure 7. Color the heavy edges with color 2 and the other edges with color 1. It is easy to check that they are rainbow connected. Thus, $rc(G) = 2$.

If $n = 4k$, let G be the graph with vertex set $X \cup Y \cup \{v\}$, where $X = \{x_1, x_2, \cdots, x_{2k-1}\}$ and $Y = \{y_1, y_2, \cdots, y_{2k}\}$ such that $N(v) = X$, X is an independent set, $G[Y]$ is a clique, and for each x_i, x_i is adjacent to every vertex of $\{y_i, y_{i+1}, \cdots, y_{i+k}\}$, where the sum is taken modulo $2k$.

We define a coloring c for the graph G by the following rules:

$$c(e) = \begin{cases} 2 & if\ e = vx_i\ for\ k+1 \leq i \leq 2k-1, \\ 2 & if\ e = x_iy_i\ for\ 1 \leq i \leq 2k-1, and\ e = x_ky_{k+1}, \\ 1 & otherwise. \end{cases}$$

Then c is a rainbow 2-coloring.

If $n = 4k+1$, construct G by adding a vertex x_{2k} to the vertex set X in the case $n = 4k$ and joining x_{2k} to every vertex of $\{v, y_{2k}, y_1, \cdots, y_{k-1}\}$. With the coloring c defined above, in addition with $c(vx_{2k}) = c(x_{2k}y_{2k}) = 2$, G is rainbow connected.

If $n = 4k+2$, construct G as follows. We add two vertices x_{2k} and y_{2k+1} to the vertex set X and Y in the case $n = 4k$, respectively. Then we join x_{2k} to v and y_{2k+1} to every vertex of Y. Finally, for each x_i, we join x_i to every vertex of $\{y_i, y_{i+1}, \cdots, y_{i+k}\}$, where the sum is taken modulo $2k+1$. With the coloring c defined above, in addition with $c(vx_{2k}) = c(x_{2k}y_{2k}) = 2$, G is rainbow connected.

If $n = 4k+3$, construct G as follows. We add two vertices x_{2k+1} and y_{2k+1} to the vertex set X and Y in the case $n = 4k+1$, respectively. Then we join x_{2k+1} to v and y_{2k+1} to every vertex in Y. For each x_i, we join x_i to every vertex of $\{y_i, y_{i+1}, \cdots, y_{i+k}\}$, where the sum is taken modulo $2k+1$. Finally, we join x_{k+1} to y_{2k+1}. With the coloring c defined above, in addition with $c(vx_{2k+1}) = c(x_{2k+1}y_{2k+1}) = 2$, G is rainbow connected.

In each of the above cases, it is easy to check that there exists a 2-coloring of $\overline{G}$ such that it is rainbow connected. Therefore, there are connected graphs G and $\overline{G}$ on n vertices such that $rc(G) = rc(\overline{G}) = 2$. ∎

Theorem 4 *For $n = 4, 5$, $rc(G) + rc(\overline{G}) \geq 6$, and for $n = 6, 7$, $rc(G) + rc(\overline{G}) \geq 5$. All these bounds are best possible.*

Proof. By Theorem 2, we have $rc(G) + rc(\overline{G}) \geq 5$ for $n \in \{4, 5, 6, 7\}$.

For $n = 4$, as we have shown, $rc(G) + rc(\overline{G}) = 6$.

For $n = 5$, the possible complementary connected graphs are shown in Figure 8.

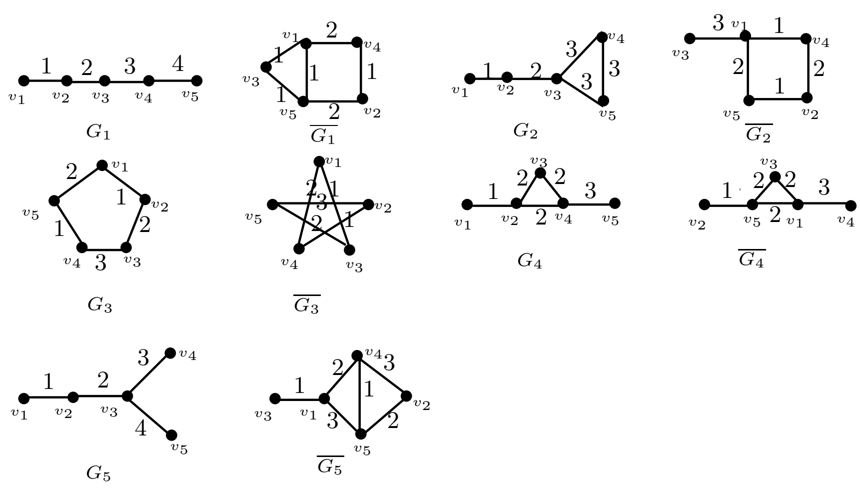

Figure 8: Complementary connected graphs for $n = 5$.

For all these cases, $rc(G) + rc(\overline{G}) \geq 6$.

For $n = 6$, let G be the cycle C_6 with vertex set $\{v_1, v_2, \cdots, v_6\}$. Then $rc(G) = 3$. We color the edges v_1v_3, v_2v_4, v_3v_5 in $\overline{G}$ by color 2, and the other edges by color 1. This coloring makes $\overline{G}$ rainbow connected. Therefore, $rc(G) + rc(\overline{G}) = 5$.

For $n = 7$, the graph G_2 in Figure 3 has rainbow connection number 2. We have shown that $rc(\overline{G_2}) \neq 2$, but we can use three colors to make it rainbow connected, just by coloring the edges v_2v_4 and v_3v_5 with color 3 and the others the same as before. It follows that $rc(\overline{G_2}) = 3$. Therefore, $rc(G) + rc(\overline{G}) = 5$. ∎

4 Concluding remark

Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* of G, if every vertex in G is at a distance at most 1 from D. Further, if D induces a connected subgraph of G, it is called a *connected dominating set* of G. The cardinality of a minimum connected dominating set in G is called its *connected dominating number*, denoted by $\gamma_c(G)$. In [3], the authors proved that for every connected graph G with minimum degree $\delta(G) \geq 2$, $rc(G) \leq \gamma_c(G) + 2$. In [6], the authors introduced a Nordhaus-Gaddum-type result for the connected dominating number. They showed that if G and $\overline{G}$ are both connected, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq n + 1$. If one uses their results, one can only get that $rc(G) + rc(\overline{G}) \leq \gamma_c(G) + \gamma_c(\overline{G}) + 4 \leq n + 5$, which is weaker than our result.

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