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ON THE RAINBOW VERTEX-CONNECTION

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Abstract

A vertex-colored graph is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. It was proved that if G is a graph of order n with minimum degree δ, then $rvc(G) < 11n/\delta$. In this paper, we show that $rvc(G) \leq 3n/(\delta+1)+5$ for $\delta \geq \sqrt{n-1}-1$ and $n \geq 290$, while $rvc(G) \leq 4n/(\delta+1)+5$ for $16 \leq \delta \leq \sqrt{n-1}-2$ and $rvc(G) \leq 4n/(\delta+1)+C(\delta)$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$. We also prove that $rvc(G) \leq 3n/4-2$ for $\delta = 3$, $rvc(G) \leq 3n/5-8/5$ for $\delta = 4$ and $rvc(G) \leq n/2-2$ for $\delta = 5$. Moreover, an example constructed by Caro et al. shows that when $\delta \geq \sqrt{n-1}-1$ and $\delta = 3, 4, 5$, our bounds are seen to be tight up to additive constants.

Keywords: rainbow vertex-connection; vertex coloring; minimum degree; 2-step dominating set.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of Bondy and Murty [2]. An edge-colored graph is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. Obviously, if G is rainbow connected, then it is also connected. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [6]. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G

rainbow connected. Observe that $diam(G) \leq rc(G) \leq n-1$. It is easy to verify that $rc(G)=1$ if and only if G is a complete graph, that $rc(G)=n-1$ if and only if G is a tree. It was shown that computing the rainbow connection number of an arbitrary graph is NP-hard [4]. There are some approaches to study the bounds of $rc(G)$, we refer to [3, 5, 10, 13].

In [10], Krivelevich and Yuster proposed the concept of rainbow vertex-connection. A vertex-colored graph is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. An easy observation is that if G is of order n then $rvc(G) \leq n-2$ and $rvc(G)=0$ if and only if G is a complete graph. Notice that $rvc(G) \geq diam(G)-1$ with equality if the diameter is 1 or 2. For rainbow connection and rainbow vertex-connection, some examples are given to show that there is no upper bound for one of parameters in terms of the other in [10]. It was also shown that computing the rainbow vertex-connection number of an arbitrary graph is NP-hard [7].

As natural combinatorial concepts, rainbow connection and rainbow vertex-connection attract many attentions of the researchers. Besides its theoretical interest, the rainbow connection can also find applications in networking problems. Actually, these new concepts come from the communication of information between agencies of government. Suppose we want to route messages in a cellular network in such a way that each link on the route between two vertices is assigned with a distinct channel. The minimum number of channels that we have to use, is exactly the rainbow connection of the underlying graph. For more details on various rainbow connections we refer the reader to a new book [11].

Krivelevich and Yuster [10] proved that if G is a graph with n vertices and minimum degree δ, then $rvc(G) < 11n/\delta$. In this paper, by the similar method of [10], we will improve this bound for given order n and minimum degree δ. We will show that $rvc(G) \leq 3n/(\delta+1)+5$ for $\delta \geq \sqrt{n-1}-1$ and $n \geq 290$, while $rvc(G) \leq 4n/(\delta+1)+5$ for $16 \leq \delta \leq \sqrt{n-1}-2$ and $rvc(G) \leq 4n/(\delta+1)+C(\delta)$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}}-2$. We also prove that $rvc(G) \leq 3n/4-2$ for $\delta=3$, $rvc(G) \leq 3n/5-8/5$ for $\delta=4$ and $rvc(G) \leq n/2-2$ for $\delta=5$.

Moreover, an example shows that when $\delta \geq \sqrt{n-1}-1$ and $\delta = 3,4,5$, our bounds are seen to be tight up to additive factors. To see this, we recall the graph constructed by Caro et al. [3], which was used to interpret the upper bound of rainbow connection $rc(G)$. A connected n-vertex graph H are constructed as follows. Take m copies of a complete graph $K_{\delta+1}$, denoted $X_1,\ldots,X_m$ and label the vertices of X_i with $x_{i,1},\ldots,x_{i,\delta+1}$. Take two copies of $K_{\delta+2}$, denoted X_0, X_{m+1} and similarly label their vertices. Now, connect $x_{i,2}$ with $x_{i+1,1}$ for $i=0,\ldots,m$ with an edge, and delete the edges $x_{i,1}x_{i,2}$ for i $=0,\ldots,m+1$.

Observe that the obtained graph H has $n = (m+2)(\delta+1)+2$ vertices, minimum degree δ and diameter $\frac{3n}{\delta+1} - \frac{\delta+7}{\delta+1}$. Therefore, the upper bound of $rvc(G)$ cannot be improved below $\frac{3n}{\delta+1} - \frac{\delta+7}{\delta+1} - 1 = \frac{3n-6}{\delta+1} - 2$.

2. $rvc(G)$ AND MINIMUM DEGREE

Let $\mathcal{G}(n, \delta)$ be the class of simple connected n-vertex graphs with minimum degree δ. Let $\ell(n, \delta)$ be the maximum value of m such that every $G \in \mathcal{G}(n, \delta)$ has a spanning tree with at least m leaves. We can obtain an trivial upper bound for $rvc(G)$.

Lemma 1. *A connected graph G of order n with maximum degree $\Delta(G)$ has $rvc(G) \leq n - \ell(n, \delta)$ and $rvc(G) \leq n - \Delta(G)$.*

Proof. It is obvious that $rvc(G) \leq n - \ell(n, \delta)$. For a connected graph G of order n with maximum degree $\Delta(G) = k$, a spanning tree T of G grown from a vertex v with degree k has maximum degree $\Delta(T) = k$. It deduces that T has at least k leaves. Thus, $rvc(G) \leq n - \Delta(G)$ holds. ■

Note that finding the maximum number of leaves in a spanning tree of G is NP-hard. Linial and Sturtevant (unpublished) [12] proved that $\ell(n, 3) \geq n/4 + 2$. For $\delta = 4$, the optimal bound $\ell(n, 4) \geq 2/5n + 8/5$ is proved in [8] and in [9]. In [8], it is also proved that $\ell(n, 5) \geq n/2 + 2$. Indeed, Kleitman and West in [9] proved that $\ell(n, \delta) \geq (1 - b \ln \delta/\delta)n$ for large δ, where b is any constant exceeding 2.5. Hence, the following theorem is obvious.

Theorem 2. *For a connected graph G of order n with minimum degree δ, $rvc(G) \leq 3n/4 - 2$ for $\delta = 3$, $rvc(G) \leq 3n/5 - 8/5$ for $\delta = 4$ and $rvc(G) \leq n/2 - 2$ for $\delta = 5$. For sufficiently large δ, $rvc(G) \leq (b \ln \delta)n/\delta$, where b is any constant exceeding* 2.5. □

Using the similar proof methods in [10], we will prove the following theorem by constructing a connected $\delta/3$-strong 2-step dominating set S whose size is at most $3n/(\delta + 1) - 2$.

Theorem 3. *A connected graph G of order n with minimum degree δ has $rvc(G) \leq 3n/(\delta + 1) + 5$ for $\delta \geq \sqrt{n-1} - 1$ and $n \geq 290$, while $rvc(G) \leq 4n/(\delta + 1) + 5$ for $16 \leq \delta \leq \sqrt{n-1} - 2$ and $rvc(G) \leq 4n/(\delta + 1) + C(\delta)$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$.*

Now we state some lemmas that are needed to prove Theorem 3. The first lemma is from [10].

Lemma 4. *If G is a connected graph with minimum degree δ, then it has a connected spanning subgraph with minimum degree δ and with less than $n(\delta + 1/(\delta+1))$ edges.* □

A set of vertices S of a graph G is called a 2-*step dominating set*, if every vertex of $V(G)\setminus S$ has either a neighbor in S or a common neighbor with a vertex in S. In [10], the authors introduced a special type of 2-step dominating set. A 2-step dominating set is k-*strong*, if every vertex that is not dominated by it has at least k neighbors that are dominated by it. We call a 2-step dominating set S is *connected*, if the subgraph induced by S is connected. Similarly, we can define the concept of *connected k-strong 2-step dominating set*. Motivated by the proof methods in [5], we can get the following result.

Lemma 5. *If G is a graph of order n with minimum degree $\delta \geq 2$, then G has a connected $\delta/3$-strong 2-step dominating set S whose size is at most $3n/(\delta+1)-2$.*

Proof. For any $T \subseteq V(G)$, denote by $N^k(T)$ the set of all vertices with distance exactly k from T. We construct a connected $\delta/3$-strong 2-step dominating set S as follows:

Procedure 1. Initialize $S' = \{u\}$ for some $u \in V(G)$. As long as $N^3(S') \neq \emptyset$, take a vertex $v \in N^3(S')$ and add vertices v, x_1, x_2 to S', where $vx_2x_1x_0$ is a shortest path from v to S' and $x_0 \in S'$.

Procedure 2. Initialize $S = S'$ obtained from Procedure 1. As long as there exists a vertex $v \in N^2(S)$ such that $|N(v) \cap N^2(S)| \geq 2\delta/3 + 1$, add vertices v, y_1 to S, where vy_1y_0 is a shortest path from v to S and $y_0 \in S$.

Clearly S' remains connected after every iteration in Procedure 1. Therefore, when Procedure 1 ends, S' is a connected 2-step dominating set. Let k_1 be the number of iterations executed in Procedure 1. Observe that when a new vertex from $N^3(S')$ is added to S', $|S' \cup N^1(S')|$ increases by at least $\delta + 1$ in each iteration. Thus, we have $k_1 + 1 \leq \frac{|S' \cup N^1(S')|}{\delta+1} = \frac{n-|N^2(S')|}{\delta+1}$. Furthermore, $|S'| = 3k_1 + 1 \leq \frac{3\left(n-|N^2(S')|\right)}{\delta+1} - 2$ since three more vertices are added in each iteration.

Notice that S also remains connected after every iteration in Procedure 2. When Procedure 2 ends, each $v \in N^2(S)$ has at most $2\delta/3$ neighbors in $N^2(S)$, i.e., has at least $\delta/3$ neighbors in $N^1(S)$, so S is a connected $\delta/3$-strong 2-step dominating set. Let k_2 be the number of iterations executed in Procedure 2. Observe that when a new vertex from $N^2(S)$ is added to S, $|N^2(S)|$ reduces by at least $2\delta/3 + 2$ in each iteration. Thus, we have $k_2 \leq \frac{|N^2(S')|}{2\delta/3+2} = \frac{3|N^2(S')|}{2\delta+6}$. Furthermore,

$$|S| = |S'| + 2k_2 \leq \frac{3\left(n - |N^2(S')|\right)}{\delta+1} - 2 + \frac{6|N^2(S')|}{2\delta+6} < \frac{3n}{\delta+1} - 2.$$

■

Before proceeding, we first recall the Lovász Local Lemma [1].

The Lovász Local Lemma Let $A_1, A_2, \ldots, A_n$ be the events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $P[A_i] \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) < 1$, then $Pr[\bigwedge_{i=1}^{n} \overline{A_i}] > 0$. □

Proof of Theorem 3. Suppose G is a connected graph of order n with minimum degree δ. By Lemma 4, we may assume that G has less than $n(\delta+1/(\delta+1))$ edges. By Lemma 5, let S be a connected $\delta/3$-strong 2-step dominating set of G with at most $3n/(\delta+1)-2$ vertices.

Suppose $\delta \geq \sqrt{n-1}-1$. Observe that each vertex v of $N^1(S)$ has less than $(\delta+1)^2$ neighbors in $N^2(S)$, since $(\delta+1)^2 \geq n-1$ and v has another neighbor in S. We assign colors to G as follows: distinct colors to each vertex of S and seven new colors to vertices of $N^1(S)$ such that each vertex of $N^1(S)$ chooses its color randomly and independently from all other vertices of $N^1(S)$. Hence, the total color we used is at most

$$|S| + 7 \leq \frac{3n}{\delta+1} - 2 + 7 = \frac{3n}{\delta+1} + 5.$$

For each vertex u of $N^2(S)$, let A_u be the event that all the neighbors of u in $N^1(S)$, denoted by $N_1(u)$, are assigned at least two distinct colors. Now we will prove $Pr[A_u] > 0$ for each $u \in N^2(S)$. Notice that each vertex $u \in N^2(S)$ has at least $\delta/3$ neighbors in $N^1(S)$ since S is a connected $\delta/3$-strong 2-step dominating set of G. Therefore, we fix a set $X(u) \subset N^1(S)$ of neighbors of u with $|X(u)| = \lceil \delta/3 \rceil$. Let B_u be the event that all of the vertices in $X(u)$ receive the same color. Thus, $Pr[B_u] \leq 7^{-\lceil \delta/3 \rceil + 1}$. As each vertex of $N^1(S)$ has less than $(\delta+1)^2$ neighbors in $N^2(S)$, we have that the event B_u is independent of all other events B_v for $v \neq u$ but at most $((\delta+1)^2-1)\lceil \delta/3 \rceil$ of them. Since for $\delta \geq \sqrt{n-1}-1$ and $n \geq 290$,

$$e \cdot 7^{\,-\lceil \delta/3 \rceil + 1} \left(((\delta+1)^2 - 1)\lceil \delta/3 \rceil + 1 \right) < 1,$$

by the Lovász Local Lemma, we have $Pr[A_u] > 0$ for each $u \in N^2(S)$. Therefore, for $N^1(S)$, there exists one coloring with seven colors such that every vertex of $N^2(S)$ has at least two neighbors in $N^1(S)$ colored differently. It remains to show that the graph G is rainbow vertex-connected. Let u, v be a pair of vertices such that $u, v \in N^2(S)$. If u and v have a common neighbor in $N^1(S)$, then we are done. Denote by x_1, y_1 and x_2, y_2, respectively, the two neighbors of u and v in $N^1(S)$ such that the colors of x_i and y_i are different for $i = 1, 2$. Without loss of generality, suppose the colors of x_1 and x_2 are also different. Indeed, there exists a required path between u and v: $ux_1w_1Pw_2x_2v$, where w_i is the neighbor of x_i in S and P is the path connecting w_1 and w_2 in S. All other cases of u, v can be checked easily.

From now on we assume $\delta \le \sqrt{n-1} - 2$. We partition $N^1(S)$ to two subsets: $D_1 = \{v \in N^1(S) :\ v \text{ has at least } (\delta+1)^2 \text{ neighbors in } N^2(S)\}$ and $D_2 = N^1(S) \setminus D_1$. Since G has less than $n(\delta + 1/(\delta+1))$ edges, we have $|D_1| \le n/(\delta+1)$. Denote by $L_1 = \{v \in N^2(S) :\ v \text{ has at least one neighbor in } D_1\}$ and $L_2 = N^2(S) \setminus L_1$.

Let $C(\delta) = 5$ for $16 \le \delta \le \sqrt{n-1} - 2$ and $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$ for $6 \le \delta \le 15$. We assign colors to G as follows: distinct colors to each vertex of $S \cup D_1$ and $C(\delta)+2$ new colors to vertices of D_2 such that each vertex of D_2 chooses its color randomly and independently from all other vertices of D_2. Hence, the total color we used is at most

$$|S| + |D_1| + C(\delta) + 2 \le \frac{3n}{\delta+1} - 2 + \frac{n}{\delta+1} + C(\delta) + 2 = \frac{4n}{\delta+1} + C(\delta).$$

For each vertex u of L_2, let A_u be the event that all the neighbors of u in D_2 are assigned at least two distinct colors. Now we will prove $Pr[A_u] > 0$ for each $u \in L_2$. Notice that each vertex $u \in L_2$ has at least $\delta/3$ neighbors in D_1. Therefore, we fix a set $X(u) \subset D_1$ of neighbors of u with $|X(u)| = \lceil \delta/3 \rceil$. Let B_u be the event that all of the vertices in $X(u)$ receive the same color. Thus, $Pr[B_u] \le (C(\delta)+2)^{-\lceil \delta/3 \rceil + 1}$. As each vertex of D_2 has less than $(\delta+1)^2$ neighbors in $N^2(S)$, we have that the event B_u is independent of all other events B_v for $v \ne u$ but at most $((\delta+1)^2 - 1)\lceil \delta/3 \rceil$ of them. Since

$$e \cdot (C(\delta)+2)^{-\lceil \delta/3 \rceil + 1} \left(((\delta+1)^2 - 1)\lceil \delta/3 \rceil + 1 \right) < 1,$$

by the Lovász Local Lemma, we have $Pr[A_u] > 0$ for each $u \in L_2$. Therefore, for D_2, there exists one coloring with $C(\delta)+2$ colors such that each vertex of L_2 has at least two neighbors in D_2 colored differently.

Similarly, we can check that the graph G is also rainbow vertex-connected in this case.

The proof is thus completed. ■

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