

3 **RAINBOW CONNECTION NUMBER OF DENSE GRAPHS**

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21 **Abstract**

22 An edge-colored graph G is rainbow connected, if any two vertices are
23 connected by a path whose edges have distinct colors. The rainbow connec-
24 tion number of a connected graph G , denoted $rc(G)$, is the smallest number
25 of colors that are needed in order to make G rainbow connected. In this
26 paper we show that $rc(G) \leq 3$ if $|E(G)| \geq \binom{n-2}{2} + 2$, and $rc(G) \leq 4$ if
27 $|E(G)| \geq \binom{n-3}{2} + 3$. These bounds are sharp.

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1. INTRODUCTION

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-colored graph G is called *rainbow connected*, if any two vertices are connected by a path whose edges have different colors. The concept of rainbow connection in graphs was introduced by Chartrand et al. in [2]. The *rainbow connection number* of a connected graph G , denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. The rainbow connection number has been studied for several graph classes. These results are presented in a recent survey [5]. Rainbow connection has an interesting application for the secure transfer of classified information between agencies, see [3].

In [4] the following problem was suggested:

Problem 1. For every $k, 1 \leq k \leq n - 1$, compute and minimize the function $f(n, k)$ with the following property: If $|E(G)| \geq f(n, k)$, then $rc(G) \leq k$.

The authors of [4] got the following results:

Proposition 2. $f(n, k) \geq \binom{n-k+1}{2} + (k - 1)$.

For convenience we repeat the proof given in [4].

Proof. We construct a graph G_k as follows: Take a $K_{n-k+1} - e$ and denote the two vertices of degree $n - k - 1$ with u_1 and u_2 . Now take a path P_k with vertices labeled $w_1, w_2, \dots, w_k$ and identify the vertices u_2 and w_1 . The resulting graph G_k has order n and size $|E(G_k)| = \binom{n-k+1}{2} + (k - 2)$. For its diameter we obtain $d(u_1, w_k) = \text{diam}(G) = k + 1$. Hence $f(n, k) \geq \binom{n-k+1}{2} + (k - 1)$. $\square$ $\blacksquare$

Moreover, in [4] $f(n, k)$ has been determined for $k = 1, 2, n - 1, n - 2$.

Proposition 3. $f(n, 1) = \binom{n}{2}, f(n, n - 1) = n - 1, f(n, n - 2) = n$.

Theorem 4. Let G be a connected graph of order $n \geq 3$. If $\binom{n-1}{2} + 1 \leq |E(G)| \leq \binom{n}{2} - 1$, then $rc(G) = 2$.

Hence $f(n, 2) = \binom{n-1}{2} + 1$. In this paper we will show that $f(n, 3) = \binom{n-2}{2} + 2$ and $f(n, 4) = \binom{n-3}{2} + 3$. One might think that equality in Proposition 1.2 always holds for any $1 \leq k \leq n$. But, as one will see, our proof technique cannot be easily used for general $k \geq 5$.

2. MAIN RESULTS

At first, we give some notation which will be used in the sequel.

Definition. Let G be a connected graph. The distance between two vertices u and v in G , denoted by $d(u, v)$, is the length of a shortest path between them in G . The distance between a vertex v and a set $S \subset V(G)$ is defined as $d(v, S) = \min_{x \in S} d(v, x)$. The k -step open neighborhood of a set $S \subset V(G)$ is defined as $N^k(S) = \{x \in V(G) \mid d(x, S) = k\}$, $k \in \{0, 1, 2, \dots\}$. When $k = 1$, we may omit the qualifier “1-step” in the above name and the superscript 1 in the notation. The neighborhood of a vertex v in $\overline{G}$, denoted by $\overline{N}(v)$, is defined as $\overline{N}(v) = \{x \mid xv \notin E(G)\}$.

We will first present a new and shorter proof for Theorem 1.4, which we restate for convenience.

Theorem 5. *Let G be a connected graph of order $n \geq 3$. If $\binom{n-1}{2} + 1 \leq |E(G)| \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

Proof. Our proof will be by induction on n . For $n = 3$, we have $f(n, n-1) = n-1 = 2 = \binom{3-1}{2} + 1$. For $n = 4$, we have $f(n, n-2) = n = 4 = \binom{4-1}{2} + 1$. So we may assume $n \geq 5$.

Since $|E(G)| \leq \binom{n}{2} - 1$, we have $1 \leq \delta(G) \leq n-2$. Choose a vertex $w \in V(G)$ with $d(w) = \delta(G)$ and set $d(w) = n-2-t$ with $0 \leq t \leq n-3$. Let $H = G - w$. Then $|E(H)| \geq \binom{n-1}{2} + 1 - d(w) = \binom{n-2}{2} + n-2+1 - (n-2-t) = \binom{n-2}{2} + 1+t = \binom{(n-1)-1}{2} + 1+t \geq \binom{n-2}{2} + 1$. Hence, H is connected; otherwise $E(H) < \binom{n-2}{2} + 1$.

Now let $\overline{N}(w) = \{v_1, v_2, \dots, v_t, v_{t+1}\}$.

Claim: $N(v_i) \cap N(w) \neq \emptyset$ for $1 \leq i \leq t+1$.

Proof of the Claim: Suppose $N(v_i) \cap N(w) = \emptyset$ for some i with $1 \leq i \leq t+1$. Then $d(v_i) \leq (t+1) - 1 = t$, thus $|E(\overline{G})| \geq |\overline{N}_H(v_i)| + |\overline{N}_G(w)| \geq (n-t-2) + (t+1) = n-1 > n-2$, a contradiction, since $|E(\overline{G})| \leq n-2$.

Hence for every vertex v_i , there is a vertex $u_i \in N(w)$ such that $u_i v_i \in E(G)$ for $1 \leq i \leq t+1$. Let H' be a subgraph of H with $V(H') = V(H)$ and $E(H') = E(H) - \{v_1 u_1, \dots, v_t u_t\}$. Then $|E(H')| \geq \binom{n-2}{2} + 1+t-t = \binom{n-2}{2} + 1 = \binom{(n-1)-1}{2} + 1$. So H' is connected, and by induction we have $rc(H') \leq 2$. Now take a 2-rainbow coloring of H' . Let $c(v_{t+1} u_{t+1}) = 1$. Then, set $c(v_i u_i) = 1$ for $1 \leq i \leq t$ and $c(e) = 2$ for all edges e which are incident with w . It is easy to check that G is 2-rainbow connected. $\square$ ■

98 In the following we give the new results of this paper.

99 **Theorem 6.** *Let G be a connected graph of order $n \geq 4$. If $|E(G)| \geq \binom{n-2}{2} + 2$,*
 100 *then $rc(G) \leq 3$.*

101 **Proof.** Our proof will be by induction on n . For $n = 4$, we have $f(n, n-1) =$
 102 $n-1 = 3 = \binom{4-2}{2} + 2$. For $n = 5$, we have $f(n, n-2) = n = 5 = \binom{5-2}{2} + 2$. So
 103 we may assume $n \geq 6$.

104 By Theorem 5, we have $rc(G) \leq 2$ for $|E(G)| \geq \binom{n-1}{2} + 1$. Hence we may
 105 assume $|E(G)| \leq \binom{n-1}{2}$. This implies $\delta(G) \leq \frac{(n-1)(n-2)}{n} = n-3 + \frac{2}{n} < n-2$.

106 Claim 1: $diam(G) \leq 3$.

107 **Proof of Claim 1:** Suppose $diam(G) \geq 4$ and consider a diameter path $v_1, v_2,$
 108 $\dots, v_{D+1}$ with $D \geq 4$. Then $d(v_1) + d(v_4) \leq n-2$ and $d(v_2) + d(v_5) \leq n-2$,
 109 implying $|E(G)| \leq \binom{n}{2} - 2(2n-3-(n-2)) = \binom{n}{2} - 2(n-1) = \binom{n-2}{2} - 1 < \binom{n-2}{2} + 2$,
 110 a contradiction.

111 Claim 2: If $\delta(G) = 1$, then $rc(G) \leq 3$.

112 **Proof of Claim 2:** Let w be a vertex with $d(w) = \delta(G) = 1$, and let $H = G - w$.
 113 Then $|E(H)| \geq \binom{n-2}{2} + 2 - 1 = \binom{n-2}{2} + 1 = \binom{(n-1)-1}{2} + 1$. Hence $rc(H) \leq 2$
 114 by Theorem 5. Take a 2-rainbow coloring for H , and set $c(e) = 3$ for the edge
 115 incident with w . Then $rc(G) \leq 3$.

116 Hence we may assume $\delta(G) \geq 2$. Let $w_1, w_2 \in V(G)$ with $w_1 w_2 \notin E(G)$.
 117 Suppose $N(w_1) \cap N(w_2) = \emptyset$. Let $H = G - \{w_1, w_2\}$. Then $|E(H)| \geq \binom{n-2}{2} +$
 118 $2 - (n-2) = \binom{n-3}{2} + 1 = \binom{(n-2)-1}{2} + 1$. Thus H is connected. Hence $rc(H) \leq 2$
 119 by Theorem 5. Consider a 2-rainbow coloring of H with colors 1 and 2. Since
 120 $diam(G) \leq 3$, there is a $w_1 w_2$ -path $w_1 u_1 u_2 w_2$. Let $c(u_1 u_2) = 1$, and then set
 121 $c(w_1 u_1) = 2, c(w_2 u_2) = 3$ and $c(e) = 3$ for all other edges incident with w_1 or w_2 .
 122 Then G is 3-rainbow connected.

123 Hence we may assume $N(w_1) \cap N(w_2) \neq \emptyset$ if $w_1, w_2 \in V(G)$ and $w_1 w_2 \notin$
 124 $E(G)$. Choose a vertex w with $d(w) = \delta(G)$ and set $d(w) = n-2-t$ with
 125 $1 \leq t \leq n-4$. As in the proof of Theorem 5, there exist vertices $u_i \in N(w)$
 126 such that $u_i v_i \in E(G)$ for $1 \leq i \leq t+1$, where $\overline{N}(w) = \{v_1, v_2, \dots, v_t, v_{t+1}\}$.
 127 Let $H = G - w$, and let H' be a subgraph of H with $V(H') = V(H)$ and
 128 $E(H') = E(H) - \{u_1 v_1, \dots, u_{t-1} v_{t-1}\}$. Then $|E(H')| \geq \binom{n-2}{2} + 2 - (n-2-t) -$
 129 $(t-1) = \binom{n-2}{2} - n + 5 = \binom{(n-1)-2}{2} + 2$.

130 Hence, if H' is connected, then by induction, H' is 3-rainbow connected. Now
 131 take a 3-rainbow coloring of H' . Let $c(u_i v_i) \in \{1, 2\}$ for $i = t, t+1$, and then set
 132 $c(u_i v_i) = 1$ for $1 \leq i \leq t-1$ and $c(e) = 3$ for all edges e incident with w . Then
 133 G is 3-rainbow connected.

134 Claim 3: If H' is disconnected, then H' has at most 2 components and one
 135 of them is a single vertex.

136 **Proof of Claim 3:** Suppose, on the contrary, that H' has $k \geq 3$ components.
 137 Let n_i be the number of vertices of the i th component. Thus $n_1 + \dots + n_k = n-1$,

138 and then

$$\begin{aligned}
|E(H')| &\leq \binom{n_1}{2} + \cdots + \binom{n_k}{2} \\
&= \sum_{i=1}^k \frac{n_i^2 - n_i}{2} = \frac{1}{2} \left(\sum_{i=1}^k n_i^2 - (n-1) \right) \\
&\leq \frac{1}{2} [1 + 1 + (n-1-2)^2 - n + 1] \\
&= \frac{1}{2} (n^2 - 7n + 12) \\
&< \binom{n-3}{2} + 2,
\end{aligned}$$

139 a contradiction. So H' has two components, that is, $n_1 + n_2 = n - 1$. If $n_1 \geq 2$,
 140 then

$$\begin{aligned}
|E(H')| &\leq \binom{n_1}{2} + \binom{n_2}{2} \\
&= \frac{n_1^2 + n_2^2 - (n-1)}{2} \\
&\leq \frac{1}{2} [2^2 + (n-3)^2 - n + 1] \\
&= \frac{1}{2} (n^2 - 7n + 14) \\
&< \binom{n-3}{2} + 2,
\end{aligned}$$

141 thus completing the proof.

142 Let $H_1 = \{v\}$, H_2 be two components of H' . We know $v \in N(w)$ (otherwise,
 143 $\delta(G) = 1$). Let $N(v) = \{w, v_1, \dots, v_{d(v)-1}\}$. Obviously, $d(v) \leq t$, and all edges
 144 $v_i v$, $1 \leq i \leq d(v) - 1$ are deleted edges. Since $|E(H_2)| = |E(H')| \geq \binom{(n-2)-1}{2} + 2$,
 145 H_2 is 2-rainbow connected by Theorem 5. Consider a 2-rainbow coloring of H_2
 146 with colors 1, 2. Set $c(vv_i) = 3$, $1 \leq i \leq d(v) - 1$, $c(wv) = 1$, $c(e) = 3$ for all
 147 other edges incident with w , $c(e) = 2$ for all other deleted edges. Then for every
 148 $x \in V(G) \setminus w$, there is a rainbow path between w and x , and for every $x \in N(w)$,
 149 there is a rainbow path vwx . For every $x \in N^2(w) \setminus N(v)$, we know $xv \notin E(G)$,
 150 and then $N(x) \cap N(v) \neq \emptyset$, which means that there exist some v_i , $1 \leq i \leq d(v) - 1$
 151 with $v_i \in N(x) \cap N(v)$, i.e., there is a rainbow path between v and x . So G is
 152 3-rainbow connected. $\square$ ■

153 **Theorem 7.** Let G be a connected graph of order $n \geq 5$. If $|E(G)| \geq \binom{n-3}{2} + 3$,
 154 then $rc(G) \leq 4$.

155 **Proof.** We apply the proof idea from the proof of Theorem 6.

156 Our proof will be by induction on n . For $n = 5$, we have $f(n, n-1) = n-1 =$
 157 $4 = \binom{5-3}{2} + 3$, and for $n = 6$, we have $f(n, n-2) = n = 6 = \binom{6-3}{2} + 3$. So we
 158 may assume $n \geq 7$.

159 By Theorem 6, we have $rc(G) \leq 3$ for $|E(G)| \geq \binom{n-2}{2} + 2$. Hence we may
 160 assume $|E(G)| \leq \binom{n-2}{2} + 1$. This implies $\delta(G) \leq \frac{(n-2)(n-3)+2}{n} = n-5 + \frac{8}{n} < n-3$.

161 Claim 4: $diam(G) \leq 4$.

162 **Proof of Claim 4:** Suppose $diam(G) \geq 5$ and consider a diameter path $v_1, v_2, \dots, v_{D+1}$
 163 with $D \geq 5$. Then $d(v_i) + d(v_{i+3}) \leq n-2$ for $i = 1, 2, 3$, implying $|E(G)| \leq$
 164 $\binom{n}{2} - 3(2n-3-(n-2)) = \binom{n}{2} - 3(n-1) = \binom{n-3}{2} - 3 < \binom{n-3}{2} + 3$, a contradiction.

165 Claim 5: If $\delta(G) = 1$, then $rc(G) \leq 4$.

166 **Proof of Claim 5:** Let w be a vertex with $d(w) = \delta(G) = 1$, and let $H = G - w$.
 167 Then $|E(H)| \geq \binom{n-3}{2} + 3 - 1 = \binom{n-3}{2} + 2 = \binom{(n-1)-2}{2} + 2$. Hence $rc(H) \leq 3$
 168 by Theorem 6. Take a 3-rainbow coloring for H , and set $c(e) = 4$ for the edge
 169 incident with w . Then $rc(G) \leq 4$.

170 Hence we may assume $\delta(G) \geq 2$.

171 Case 1 : There are $w_1, w_2 \in V(G)$, $w_1 w_2 \notin E(G)$, with $N(w_1) \cap N(w_2) = \emptyset$
 172 and $d(w_1) + d(w_2) \leq n-3$.

173 Let $H = G - \{w_1, w_2\}$. Then $|E(H)| \geq \binom{n-3}{2} + 3 - (n-3) = \binom{n-4}{2} + 2 =$
 174 $\binom{(n-2)-2}{2} + 2$. We claim that H is connected. Otherwise, by the proof of Theorem
 175 6, we know that H has at most 2 components and one of them is a single vertex.
 176 Thus $\delta(G) = 1$, a contradiction. Then $rc(H) \leq 3$ by Theorem 6. Consider a
 177 3-rainbow coloring of H with colors 1, 2, 3. If there is a rainbow path $P = xyz$
 178 of length 2 between $N(w_1)$ and $N(w_2)$, where $x \in N(w_1), z \in N(w_2)$, then let
 179 $c(xy) = 1, c(yz) = 2$ and set $c(w_1 x) = 3, c(w_2 z) = 4$ and $c(e) = 4$ for all other
 180 edges incident with w_1 or w_2 . Then G is 4-rainbow connected. If all paths of
 181 length 2 between $N(w_1)$ and $N(w_2)$ are not rainbow, then we choose a path
 182 $P = xyz$, where $x \in N(w_1), z \in N(w_2)$. Let $c(xy) = c(yz) = 1$, and then keep
 183 the colors of all the edges in $E(H)$ except for yz . Then set $c(yz) = 4, c(w_1 x) =$
 184 $2, c(w_2 z) = 3$ and $c(e) = 4$ for all other edges incident with w_1 or w_2 . It is only
 185 need to check that G is 4-rainbow connected. Since $\delta \geq 2$, then there exists a v
 186 such that $c(w_1 v) = 4$. For every $w \in V(G) \setminus N(w_1)$, there is a rainbow path P
 187 from w_1 to w not containing yz . Otherwise, there is a rainbow path of length 2
 188 between $N(w_1)$ and $N(w_2)$, and so $w_1 v P w$ is a rainbow path. For w_2 , the proof
 189 is similar.

190 Case 2 : For all $w_1, w_2 \in V(G)$, $w_1 w_2 \notin E(G)$, we have $N(w_1) \cap N(w_2) \neq \emptyset$
 191 or $d(w_1) + d(w_2) \geq n-2$.

192 We know that in this case $diam(G) \leq 3$. Choose a vertex w with $d(w) =$
 193 $\delta(G)$, and set $d(w) = n-2-t$ with $2 \leq t \leq n-4$.

194 Subcase 2.1 : $N^3(w) = \emptyset$.

195 As in the proof of Theorem 5, there exist vertices $u_i \in N(w)$ such that

196 $u_i v_i \in E(G)$ for $1 \leq i \leq t+1$, where $\overline{N}(w) = \{v_1, v_2, \dots, v_t, v_{t+1}\}$. Let $H =$
 197 $G - w$, and let H' be a subgraph of H with $V(H') = V(H)$ and $E(H') =$
 198 $E(H) - \{u_1 v_1, \dots, u_{t-2} v_{t-2}\}$. Then $|E(H')| \geq \binom{n-3}{2} + 3 - (n-2-t) - (t-2) =$
 199 $\binom{n-3}{2} - n + 7 = \binom{(n-1)-3}{2} + 3$.

200 If H' is connected, then by induction, H' is 4-rainbow connected. Now take
 201 a 4-rainbow coloring of H' . Let $c(u_i v_i) \in \{1, 2, 3\}$ for $i = t-1, t, t+1$. Then set
 202 $c(u_i v_i) = 1$ for $1 \leq i \leq t-2$ and $c(e) = 4$ for all edges e incident with w . Then
 203 G is 4-rainbow connected.

204 If H' is disconnected, we claim that H' has at most 3 components. Otherwise,
 205 $|E(H')| < \binom{n-4}{2} + 3$. If H' has exactly 3 components H_1, H_2, H_3 , we may assume
 206 that $|H_3| \geq |H_2| \geq |H_1| \geq 1$, $|H_1| + |H_2| + |H_3| = n-1$. If $|H_2| \geq 2$, then
 207 $|E(H')| \leq \binom{|H_1|}{2} + \binom{|H_2|}{2} + \binom{|H_3|}{2} \leq 1 + \binom{n-4}{2} < \binom{n-4}{2} + 3$. So $|H_1| = |H_2| = 1$, and
 208 let $V(H_1) = \{u_1\}$, $V(H_2) = \{u_2\}$ and $u_1, u_2 \in N(w)$. Then $|E(H_3)| \geq \binom{n-4}{2} + 3 \geq$
 209 $\binom{(n-3)-1}{2} + 3$. Hence, by Theorem 5, H_3 is 2-rainbow connected. Now consider a
 210 2-rainbow coloring of H' with colors 1, 2. Set $c(wu_1) = 1, c(wu_2) = 2, c(e) = 4$ for
 211 all the edges e incident with w , and set $c(f) = 3$ for all edges f incident with u_1
 212 or u_2 except for wu_1, wu_2 , as well as $c(g) = 1$ for all other deleted edges g . Then
 213 G is 4-rainbow connected.

214 If H' has exactly 2 components H_1, H_2 . we may assume that $|H_2| \geq |H_1| \geq 1$.
 215 First, $|H_1| = 1$, and let $V(H_1) = \{u_1\}$ and $u_1 \in N(w)$. Then $|E(H_2)| \geq \binom{n-4}{2} +$
 216 $3 \geq \binom{(n-2)-2}{2} + 3$. Hence, by Theorem 6, H_2 is 3-rainbow connected. Now consider
 217 a 3-rainbow coloring of H' with colors 1, 2, 3. Set $c(wu_1) = 1, c(e) = 4$ for all edges
 218 e incident with w or u_1 except for wu_1 and set $c(g) = 2$ for all other deleted edges
 219 g . Then G is 4-rainbow connected. Second, $|H_1| \geq 2$. Since $n \geq 7$, then $|H_2| \geq 3$.
 220 Thus if $|H_1| \geq 3$, we have

$$\begin{aligned}
 |E(H_1)| &\geq \binom{n-4}{2} + 3 - \binom{|H_2|}{2} \\
 &= \frac{1}{2} [|H_1|^2 - 3|H_1| + 4] + |H_1||H_2| - 3|H_2| - 2|H_1| + 7 \\
 &\geq \binom{|H_1|-1}{2} + 1 + 3(n-4) - 3(n-1) + |H_1| + 7 \\
 &\geq \binom{|H_1|-1}{2} + 1
 \end{aligned}$$

221 Similarly, $|E(H_2)| \geq \binom{|H_2|-1}{2} + 1$. Obviously if $|H_1| = 2$, H_1, H_2 are 2-
 222 rainbow connected. Hence when $|H_1| \geq 2$, both H_1, H_2 are 2-rainbow connected.
 223 Consider a 2-rainbow coloring of H' with colors 1, 2. Set $c(wv) = 4$ for all $v \in$
 224 $V(H_1)$, $c(wv) = 3$ for all $v \in V(H_2)$, $c(uv) = 4$ for all $u \in V(H_2) \cap N(w), v \in$
 225 $V(H_1) \cap N^2(w)$, $c(uv) = 3$ for all $u \in V(H_1) \cap N(w), v \in V(H_2) \cap N^2(w)$, $c(e) =$
 226 1 for all other edges e . Then G is 4-rainbow connected.

227 Subcase 2.2 : $N^3(w) \neq \emptyset$.

228 For every $u \in N^3(w)$, $wu \notin E(G)$ and $N(w) \cap N(u) = \emptyset$, then $d(w) + d(u) =$
 229 $n-2$, that is, $N(u) = N^2(w) \cup N^3(w) \setminus \{u\}$. Let $N(w) = \{u_1, \dots, u_{n-t-2}\}$, $N^2(w) =$
 230 $\{v_1, \dots, v_p\}$, $p \geq 1$, $N^3(w) = \{v_{p+1}, \dots, v_{t+1}\}$.

If $p = 1$, v_1 is a cut vertex and $G[N^2(w) \cup N^3(w)]$ is a complete graph. Let H_1, H_2 be two blocks of $G - v_1$, we may assume that H_2 is a complete graph. Let $N_{H_1}(v_1) = \{u_1, \dots, u_s\}$, $1 \leq s \leq n - t - 2$. Then $K_{2,s}$ is a spanning subgraph of $G[w, v_1, u_1, \dots, u_s]$. If $s \geq 2$, then $K_{2,s}$ is 4-rainbow connected. Now we give a 4-coloring of $K_{2,s}$ as follows:

$$c(e) = \begin{cases} j+1, & \text{if } e = u_i w \text{ if } i \in \{3j+1, 3j+2, 3j+3\} \text{ for } 0 \leq j \leq 2, \\ 4, & \text{if } e = u_i w \text{ for } i > 9, \\ i \bmod 3, & \text{if } e = v_1 u_i \text{ for } i \leq 9, \\ 3, & \text{if } e = v_1 u_i \text{ for } i > 9. \end{cases}$$

231 For every $u_k (s < k \leq n - t - 2)$, $u_k v_j \notin E(G)$ and $N(u_k) \cap N(v_j) = \emptyset$ for
 232 $2 \leq j \leq t+1$, then $N(u_k) = N(w) \cup \{w\} \setminus \{u_k\}$. Set $c(u_k u_j) = c(w u_j)$ for
 233 $1 \leq j \leq s$, $c(e) = 1$ for all other edges e in $E(H_1)$, $c(e) = 4$ for $e \in E(H_2)$. Then
 234 G is 4-rainbow connected. If $s = 1$, then G is 3-rainbow connected.

235 If $p = 2$, let $H_1 = G[w \cup N(w) \cup N^2(w)]$, $H_2 = G[N^2(w) \cup N^3(w)]$, then
 236 $|H_1| + |H_2| = n + 2$, $|H_1| \geq 5$, $|H_2| \geq 3$. If $|H_2| = 3$, then $d(v_3) = 2 = d(w)$, thus
 237 $n = 6$. If $|H_2| = 4$, $n = 7$, set $c(w u_1) = 4$, $c(w u_2) = 3$, $c(u_1 v_1) = 2$, $c(u_2 v_2) =$
 238 1 , $c(e) = 1$ for all $e \in E(H_2)$, then G is 4-rainbow connected. If $|H_2| = 4$, $n \geq 8$,
 239 $|E(H_1)| \geq \binom{n-3}{2} + 3 - 5 = \binom{(n-2)-2}{2} + n - 6$, then H_1 is 3-rainbow connected.
 240 Consider a 3-rainbow coloring of H_1 with 2, 3, 4. Set $c(e) = 1$ for all $e \in E(H_2)$,
 241 then G is 4-rainbow connected. If $|H_2| \geq 5$, $|H_1| \geq 6$, then

$$\begin{aligned} |E(H_1)| &\geq \binom{n-3}{2} + 3 - \binom{|H_2|}{2} \\ &= \frac{1}{2} [|H_1|^2 - 5|H_1| + 6] + 2 + |H_1||H_2| - 3|H_1| - 5|H_2| + 13 \\ &\geq \binom{|H_1| - 2}{2} + 2 + 5(n + 2 - 5) - 5(n + 2) + 2|H_1| + 13 \\ &\geq \binom{|H_1| - 2}{2} + 2. \end{aligned}$$

242 Hence H_1 is 3-rainbow connected. Consider a 3-rainbow coloring of H_1 with colors
 243 2, 3, 4. Set $c(e) = 1$ for all $e \in E(H_2)$, then G is 4-rainbow connected. When
 244 $|H_2| \geq 5$, $|H_1| = 5$, set $c(w u_1) = 4$, $c(w u_2) = 3$, $c(u_1 v_1) = 2$, $c(u_2 v_2) = 1$, $c(e) = 1$
 245 for all $e \in E(H_2)$, then G is 4-rainbow connected.

246 Now we may assume that $p \geq 3$. For every $v_i \in N^2(w)$, there is a vertex $u_i \in$
 247 $N(w)$ such that $u_i v_i \in E(G)$. Let H be the graph be deleting w and edges $u_i v_i$ for

248 $v_i \in N^2(w) \setminus \{v_1, v_2, v_3\}$ and edges v_1v_i for $p+1 \leq i \leq t+1$, then $|E(H)| \geq \binom{n-4}{2} +$
 249 3. If H is connected, then by induction, H is 4-rainbow connected. Consider a
 250 4-rainbow coloring of H with colors 1, 2, 3, 4. Let $c(u_iv_i) \in \{1, 2, 3\}$ for $i = 1, 2, 3$.
 251 We may assume that $c(u_1v_1) = 1$. Set $c(e) = 4$ for all edges e incident with w ,
 252 $c(v_1v_i) = 2, p+1 \leq i \leq t+1$, $c(v_iu_i) = 3$ for all other edges between $N(w)$ and
 253 $N^2(w)$. Then G is 4-rainbow connected.

254 If H is disconnected, similar to the proof in Subcase 2.1, H has at most 3
 255 components. If H has exactly 3 components with two single vertices in $N(w)$,
 256 denoted by u, u' , then H_3 is 2-rainbow connected. Consider a 2-rainbow coloring
 257 of H_3 with colors 1, 2. Let $c(u_1v_1) = 1$, set $c(wu) = c(wu') = 3, c(e) = 4$ for all
 258 edges e incident with w , $c(v_1v_i) = 2, p+1 \leq i \leq t+1$, $c(f) = 3$ for all edges f
 259 incident with u except for wu , $c(g) = 4$ for all edges g incident with u' except for
 260 wu' , $c(h) = 1$ for all the remaining edges h . Then G is 4-rainbow connected.

261 If H has exactly two components H_1, H_2 . First, $|H_1| = 1$, let $V(H_1) = \{u\} \subseteq$
 262 $N(w)$, then H_2 is 3-rainbow connected by the proof of Subcase 2.1. Now consider
 263 a 3-rainbow coloring of H_2 with colors 1, 2, 3. Set $c(wu) = 1, c(e) = 4$ for all edges
 264 e incident with w or u except for wu , $c(f) = 1$ for all the remaining edges f . Then
 265 G is 4-rainbow connected. Second, $|H_1| \geq 2$, then both H_1, H_2 are 2-rainbow
 266 connected by the proof of Subcase 2.1. We may assume that H_2 contains $N^3(w)$.
 267 Now consider a 2-rainbow coloring of H_1, H_2 with colors 1, 2. Set $c(wv) = 3$ for
 268 all $v \in V(H_1)$, $c(wv) = 4$ for all $v \in V(H_2)$ $c(v_1v_i) = 4, p+1 \leq i \leq t+1$, $c(e) = 3$
 269 for all the remaining edges e . Then G is 4-rainbow connected.

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