Note on minimally d-rainbow connected graphs *

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Abstract

An edge-colored graph G, where adjacent edges may have the same color, is *rainbow connected* if every two vertices of G are connected by a path whose edges have distinct colors. A graph G is *d-rainbow connected* if one can use d colors to make G rainbow connected. For integers n and d let $t(n,d)$ denote the minimum size (number of edges) in d-rainbow connected graphs of order n. Schiermeyer got some exact values and upper bounds for $t(n,d)$. However, he did not present a lower bound of $t(n,d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$. In this paper, we improve his lower bound of $t(n,2)$, and get a lower bound of $t(n,d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to book [2] for notation and terminology not described here. A path in an edge-colored graph G, where adjacent edges may have the same color, is a *rainbow path* if no pair of edges is colored the same. A *d-rainbow connected graph G* has a d-edge-coloring such that any two vertices are connected by a rainbow path. Note that Schiermeyer used the term *rainbow d-connected* in [5]. However we think that it is better to use the term *d-rainbow connected* since this will distinguish it from the term *rainbow d-connectivity*, which means that there are many rainbow paths between every pair of vertices, see [4].

The rainbow connection number $rc(G)$ of G is the minimum integer d such that G has a d-rainbow connected coloring (see [4]). It is easy to see that

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$rc(G) \geq diam(G)$ for any connected graph G, where $diam(G)$ is the diameter of G.

For integers n and d let $t(n, d)$ denote the minimum size (number of edges) in d-rainbow connected graphs of order n. Because a network which satisfies our requirements and has as few links as possible can cut costs, reduce the construction period and simplify later maintenance, the study of this parameter is significant. Schiermeyer [5] mainly investigated the upper bound of $t(n, d)$ and showed the following results.

Theorem 1. (Schiermeyer[5])
(i) $t(n, 1) = \binom{n}{2}$.
(ii) $t(n, 2) \leq (n + 1)\lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor} - 2$.
(iii) $t(n, 3)) \leq 2n - 5$.
(iv) For $4 \leq d < \frac{n-1}{2}$, $t(n, d) \leq n - 1 + \left\lceil \frac{n-2}{d-2} \right\rceil$.
(v) For $\frac{n}{2} \leq d \leq n - 2$, $t(n, d) = n$.
(vi) $t(n, n - 1) = n - 1$.

In [5], Schiermeyer also got a lower bound of $t(n, 2)$ by an indirect method, and showed that $t(n, 2) \geq n \log_2 n - 4n \log_2 \log_2 n - 5n$ for sufficiently large n. Nevertheless, he did not present a lower bound of $t(n, d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$. In this paper, we use a different method to improve his lower bound of $t(n, 2)$, and moreover, we get a lower bound of $t(n, d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$.

2 Main results

Let G be a 2-rainbow connected graph of order n with maximum degree $\Delta(G)$. Pick a vertex $u \in V(G)$. Since $d(u) \leq \Delta(G)$, there exist at most $\Delta(G)$ vertices adjacent to u, and at most $\Delta(G)(\Delta(G) - 1)$ vertices at distance 2 from u. Since $diam(G) \leq rc(G) \leq 2$, we derive $n \leq 1 + \Delta(G) + \Delta(G)(\Delta(G) - 1)$. Thus, $\Delta(G) \geq \sqrt{n-1}$. Since $\Delta(G)$ is an integer, we get

$$\Delta(G) \geq \left\lceil \sqrt{n-1} \right\rceil. \tag{1}$$

Next, we investigate the lower bound of $t(n, d)$.

Proposition 1. *For sufficiently large* n, $t(n, 2) \geq n \log_2 n - 4n \log_2 \log_2 n - 2n$.

Proof. Let G be a graph with diameter 2 and c be a 2-rainbow connected coloring of G with colors blue and red. Set $k = \lfloor \log_2 n \rfloor^2 - 1$ and denote by S the set of vertices with degrees less than k. Assume that $S = \{u_1, u_2, \ldots, u_s\}$ and $\bar{S} = V(G) \backslash S = \{u_{s+1}, u_{s+2}, \ldots, u_{s+\ell}\}$, where $s + \ell = n$. For sufficiently large n, $k = \lfloor \log_2 n \rfloor^2 - 1 \leq \left\lceil \sqrt{n-1} \right\rceil \leq \Delta(G)$. By (1) we know that $\bar{S}$ is nonempty. If $\ell = |\bar{S}| \geq \frac{2n}{\log_2 n}$, then

$$\begin{aligned} e(G) \geq \frac{1}{2}\sum_{v\in \bar{S}} d_G(v) &\geq \frac{2n}{2\log_2 n}\left(\lfloor \log_2 n\rfloor^2 - 1\right) \\ &\geq \frac{n}{\log_2 n}\left((\log_2 n - 1)^2 - 1\right) \\ &= n\log_2 n - 2n, \end{aligned}$$

and we are done.

Suppose $\ell < \frac{2n}{\log_2 n}$, that is, $s > n - \frac{2n}{\log_2 n}$. It is sufficient to show that $e(S, \bar{S}) \geq n\log_2 n - 4n\log_2\log_2 n - 2n$.

For every u_i, $1 \leq i \leq s$, we define a vector as follows:

$$\vec{b}(u_i) = (b_{i,1}, b_{i,2}, \ldots, b_{i,\ell}),$$

where

$$b_{i,j} = \begin{cases} 1, & if\ c(u_iu_{s+j})\ is\ red; \\ -1, & if\ c(u_iu_{s+j})\ is\ blue; \\ 0, & if\ u_i\ and\ u_{s+j}\ is\ nonadjacent. \end{cases}$$

Suppose $|N(u_i) \cap \bar{S}| = a_i$, where $1 \leq i \leq s$. Then $e(S, \bar{S}) = \sum_{i=1}^{s} a_i$. We now estimate the value of $e(S, \bar{S})$. For each $\vec{b}(u_i)$, we define a set B_i as follows: $B_i = \{$vectors obtained from $\vec{b}(u_i)$ by replacing "0" of $\vec{b}(u_i)$ by "1" or "-1"$\}$. Because $|N(u_i) \cap \bar{S}| = a_i$, we have $|B_i| = 2^{\ell - a_i}$ for each i, where $1 \leq i \leq s$. Set $B = \bigcup_{i=1}^{s} B_i$. Then B is a multiset of ℓ-dimensional vectors with elements 1 and -1. For each $\vec{b} \in B$, $n_{\vec{b}}$ denotes the number of $\vec{b}$'s in B. We have the following claim.

Claim 1. For each $\vec{b} \in B$, $n_{\vec{b}} \leq k^2$.

Proof of Claim 1. If Claim 1 is not true, that is, there exists a vector $\vec{b}$, without loss of generality, say $\vec{b} = (b_1, b_2, \ldots, b_\ell)$, such that $n_{\vec{b}} \geq k^2 + 1$. Clearly, it is not possible that there exists some B_i such that B_i contains two $\vec{b}$'s. Thus, there exist $k^2 + 1$ integers, without loss of generality, say $1, 2, \ldots, k^2 + 1$, such that $\vec{b} \in B_i$, where $1 \leq i \leq k^2 + 1$. We next show that for each j, the distance between u_1 and u_j in $G[S]$ is at most 2, where $2 \leq j \leq k^2 + 1$. In fact, suppose that $u_1u_{s+r}, u_ju_{s+r} \in E(G)$. But, since $c(u_1u_{s+r}) = b_r = c(u_ju_{s+r})$ for each r, there exists no rainbow path between u_1 and u_j through a vertex of T. Hence, there must exist a rainbow path between u_1 and u_j with length at most 2 in $G[S]$. On one hand, since the distance between u_1 and u_j in $G[S]$ is at most 2 for each $2 \leq j \leq k^2 + 1$, we have $|N_{G[S]}(u_1) \cup N^2_{G[S]}(u_1)| \geq k^2$, where $N^2_{G[S]}(u_1)$ is a set of vertices at distance 2 from u_1 in $G[S]$. On the other hand, since $\Delta(G[S]) < k$, $|N_{G[S]}(u_1) \cup N^2_{G[S]}(u_1)| \leq (k-1) + (k-1)(k-2) = (k-1)^2$, a contradiction. Thus the claim is true. $\square$

By Claim 1 we know

$$\sum_{i=1}^{s} |B_i| \leq k^2 2^\ell,$$

Since $|B_i| = 2^{\ell - a_i}$ for each i, where $1 \leq i \leq s$,

$$\sum_{i=1}^{s} 2^{-a_i} \leq k^2.$$

By the inequality between the geometrical and arithmetical means, we have

$$\sqrt[s]{2^{-e(S,\bar{S})}} = \sqrt[s]{2^{-\sum_i^s a_i}} \leq \frac{1}{s}\sum_{i=1}^{s} 2^{-a_i} \leq \frac{k^2}{s}.$$

Using the log function on both sides, we get

$$e(S,\bar{S}) \geq s\log_2 s - s\log_2 k^2. \tag{2}$$

Note that $k^2 = (\lfloor \log_2 n \rfloor^2 - 1)^2 \leq ((\log_2 n)^2 - 1)^2 \leq (\log_2 n)^4 - 2(\log_2 n)^2 + 1 \leq (\log_2 n)^4$ and so

$$e(S,\bar{S}) \geq s\log_2 s - 4s\log_2\log_2 n. \tag{3}$$

Since $e(S,\bar{S})$ is monotonically increasing in s and $s > n - \frac{2n}{\log_2 n}$, we have

$$\begin{aligned}
e(S,\bar{S}) &\geq \left(n - \frac{2n}{\log_2 n}\right)\log_2\left(n - \frac{2n}{\log_2 n}\right) - 4\left(n - \frac{2n}{\log_2 n}\right)\log_2\log_2 n \\
&= n\log_2\left(n - \frac{2n}{\log_2 n}\right) - \frac{2n}{\log_2 n}\log_2\left(n - \frac{2n}{\log_2 n}\right) \\
&\quad -4n\log_2\log_2 n + \frac{8n}{\log_2 n}\log_2\log_2 n \\
&= n\log_2 n + n\log_2\left(1 - \frac{2}{\log_2 n}\right) - 2n - \frac{2n}{\log_2 n}\log_2\left(1 - \frac{2}{\log_2 n}\right) \\
&\quad -4n\log_2\log_2 n + \frac{8n}{\log_2 n}\log_2\log_2 n \\
&\geq n\log_2 n - 4n\log_2\log_2 n - 2n + n\log_2\left(1 - \frac{2}{\log_2 n}\right) + \frac{8n}{\log_2 n}\log_2\log_2 n \\
&= n\log_2 n - 4n\log_2\log_2 n - 2n + \frac{2n}{\log_2 n}\log_2\left(\left(1 - \frac{2}{\log_2 n}\right)^{\frac{\log_2 n}{2}}(\log_2 n)^4\right) \\
&\geq n\log_2 n - 4n\log_2\log_2 n - 2n.
\end{aligned}$$

The last inequality holds since $\left(1 - \frac{2}{\log_2 n}\right)^{\frac{\log_2 n}{2}}$ is monotonically increasing in n and tends to $\frac{1}{e}$. This completes the proof. □

Before showing the lower bound on $t(n, d)$ for each $d \geq 3$, we need the following theorem and observation.

Theorem 2. (Jarry and Laugier[3])

Any 2-edge-connected graph of order n and of odd diameter $p \geq 2$ contains at least $\left\lceil \frac{np-(2p+1)}{p-1} \right\rceil$ edges. Any 2-edge-connected graph of order n and of even diameter p contains at least $\min\left\{ \left\lceil \frac{np-(2p+1)}{p-1} \right\rceil, \left\lceil \frac{(n-1)(p+1)}{p} \right\rceil \right\}$ edges.

Note that $\left\lceil \frac{np-(2p+1)}{p-1} \right\rceil = \left\lceil \frac{(n-1)(p-1)+n+p-1-(2p+1)}{p-1} \right\rceil = \left\lceil n-1+\frac{n-p-2}{p-1} \right\rceil \geq \left\lceil n-1+\frac{n-p-2}{p} \right\rceil = n-2+\left\lceil \frac{n-2}{p} \right\rceil$ and $\left\lceil \frac{(n-1)(p+1)}{p} \right\rceil = \left\lceil \frac{(n-1)p+n-1}{p} \right\rceil = \left\lceil n-1+\frac{n-1}{p} \right\rceil \geq \left\lceil n-1+\frac{n-p-2}{p} \right\rceil = n-2+\left\lceil \frac{n-2}{p} \right\rceil$. Thus, any 2-edge-connected graph of order n and of diameter $p \geq 2$ contains at least $n-2+\left\lceil \frac{n-2}{p} \right\rceil$ edges.

Let G be a graph and c be a rainbow connected coloring of G. It is easy to see that different bridges of G must receive different colors under c. Therefore, the following observation is obvious.

Observation 1. *The rainbow connection number of a graph is at least the number of bridges of the graph.*

Proposition 2. *For $3 \leq d < \left\lceil \frac{n}{2} \right\rceil$,*

$$t(n, d) \geq n - d - 3 + \left\lceil \frac{n-1}{d} \right\rceil.$$

Proof. Let G be a k-rainbow connected graph of order n. Suppose that G has k bridges and G' is the graph obtained from G by deleting all the bridges. Then G' has $k+1$ components. We have $k \leq d$ by Observation 1. Suppose that $G_1, G_2, \cdots, G_{k_1}$ are the nontrivial components of G'. Thus, G' has $k_2 = k+1-k_1$ trivial components. Let n_i denote the order of G_i and d_i denote the diameter of G_i. We have that $n_1 + n_2 + \cdots + n_{k_1} = n - k_2$.

Claim 2. Each of the graphs G_i is either a 2-edge-connected graph with diameter at least 2 or a complete graph of order at least 3, where $1 \leq i \leq k_1$.

Proof of Claim 2. Suppose that some of the graphs G_i is neither a 2-edge-connected graph with diameter at least 2, nor a complete graph of order at least 3. That is, the graph G_i is a complete graph of order less than 3. Since G_i is nontrivial, G_i is a complete graph of order 2. However, the only edge of G_i is clearly a bridge of G, a contradiction. Thus, this claim holds.

Now consider the number of edges of G_i. If G_i is a 2-edge-connected graph with diameter $d_i \geq 2$, then by Theorem 2 we have that $e(G_i) \geq n_i - 2 + \left\lceil \frac{n_i-2}{d_i} \right\rceil$. If not, that is, G_i is a complete graph of order at least 3 by Claim 2 and we have that $e(G_i) \geq \binom{n_i}{2} \geq n_i - 2 + \left\lceil \frac{n_i-2}{d_i} \right\rceil$. Thus, $e(G_i) \geq n_i - 2 + \left\lceil \frac{n_i-2}{d_i} \right\rceil$ for each i, where $1 \leq i \leq k_1$.

Claim 3. $d_i \le d$.

Proof of Claim 3. Let x and y be two vertices in G_i. Since the shortest path connecting x and y in G_i must be a path contained G_i, we have $d_{G_i}(x, y) \le d_G(x, y)$. Thus, $d_i \le diam(G) \le d$.

Now evaluate the number of edges in G. We have

$$\begin{aligned}
e(G) &= k + \sum_{i=1}^{k_1} e(G_i) \\
&= k + \sum_{i=1}^{k_1} \left(n_i - 2 + \left\lceil \frac{n_i - 2}{d_i} \right\rceil\right) \\
&\ge k + \sum_{i=1}^{k_1} \left(n_i - 2 + \left\lceil \frac{n_i - 2}{d} \right\rceil\right) \\
&\ge k + \sum_{i=1}^{k_1} \left(n_i - 2 + \frac{n_i - 2}{d}\right) \\
&= k + (n - k_2) - 2k_1 + \frac{(n - k_2) - 2k_1}{d} \\
&= n - 1 - k_1 + \frac{n - k - 1 - k_1}{d} \\
&\ge n - 1 - k + \frac{n - 2k - 1}{d} \\
&\ge n - d - 1 + \frac{n - 2d - 1}{d} \\
&= n - d - 3 + \frac{n - 1}{d}.
\end{aligned}$$

Thus, $t(n, d) \ge n - d - 3 + \lceil \frac{n-1}{d} \rceil$. □

In the recent paper [1] the upper bound of Schiermeyer in [5] is improved to $t(n, d) \le \frac{d(n-2)}{d-1}$ for $3 \le d \le n - 1$. Moreover, equality for $d = 3$ is proved in [1].

Combining the above results, we have the following theorem.

Theorem 3. (i) *For sufficiently large n, $t(n, 2) \ge n \log_2 n - 4n \log_2 \log_2 n - 2n$.*

(ii) *For $3 \le d < \lceil \frac{n}{2} \rceil$, $n - d - 3 + \left\lceil \frac{n-1}{d} \right\rceil \le t(n, d) \le \frac{d(n-2)}{d-1}$.*

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