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On λ-Fold Equipartite Oberwolfach Problem with Uniform Table Sizes

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Abstract. We consider the following generalization of the Oberwolfach problem: "At a gathering there are n delegations each having m people. Is it possible to arrange a seating of mn people present at s round tables $T_1, T_2, \ldots, T_s$ (where each T_i can accommodate $t_i \geq 3$ people and $\sum t_i = mn$) for k different meals so that each person has every other person not in the same delegation for a neighbor exactly λ times?" For $\lambda = 1$, Liu has obtained the complete solution to the problem when all tables accommodate the same number t of people. In this paper, we give the complete solution to the problem for $\lambda \geq 2$ when all tables have uniform sizes t.

Keywords: resolvable cycle design, Hamiltonian decomposition, cycle frame, cycle factorization

1. Introduction

Throughout the paper, we use C_n for a cycle on n vertices, P_n for a path on n vertices, K_n for the complete graph on n vertices, and $K(m:n)$ for the complete n-partite graph with m vertices in each partite set (also called complete equipartite graph), namely, $K(m:n) = K(m_1, m_2, \ldots, m_n)$ with $m_1 = m_2 = \cdots = m_n = m$. Also, for a graph G, we use λG to represent the multi-graph obtained from G by replacing each edge of G with λ copies of it.

A *factor* F of a graph G is a subgraph for which $V(F) = V(G)$. An r-*factor* of G is a factor that is regular of degree r. Clearly, a 2-factor is a disjoint union of cycles. An r-*factorization* of a graph G is a partition of the edge set $E(G)$ into r-*factors*. Thus, a graph G having a 2-factorization must be regular of even degree. An $\{H_1, H_2, \ldots, H_k\}$-factorization of a graph G is a partition of the edge set $E(G)$ into factors such that each component of any factor is isomorphic to H_i for some $1 \leq i \leq k$. In particular, an H-*factorization* of G is a partition of $E(G)$ into factors such that each factor is a disjoint union of H's. Consequently, a C_t-factorization of G is a 2-factorization with each 2-factor being a disjoint union of C_t's.

The well-known Oberwolfach problem [3] was first formulated by Ringel in 1967: "Is it possible to seat an odd number n of people at s round tables $T_1, T_2, \ldots, T_s$ (where each T_i can accommodate $t_i \geq 3$ people and $\sum t_i = n$) for k different meals so that each person has every other person for a neighbor exactly once?" In terms of graph theory, this problem is equivalent to asking for an odd integer n, is it possible for K_n to have a 2-factorization in which each 2-factor consists of cycles of lengths $t_1, t_2, \ldots, t_s$?

In [5], Huang, Kotzig, and Rosa considered the following "spouse-avoiding" variant of the Oberwolfach problem: "At a gathering there are n couples. Is it possible to arrange a seating of $2n$ people present at s round tables $T_1, T_2, \ldots, T_s$ (where each T_i can accommodate $t_i \geq 3$ people and $\sum t_i = mn$) for k different meals so that each person has every other person except his or her spouse for a neighbor exactly once?" In terms of graph theory, this problem is equivalent to asking for a 2-factorization of $K(2:n)$ in which each 2-factor consists of cycles of lengths $t_1, t_2, \ldots, t_s$?

Here we consider the following generalization of the Oberwolfach problem: "At a gathering there are n delegations each having m people. Is it possible to arrange a seating of mn people present at s round tables $T_1, T_2, \ldots, T_s$ (where each T_i can accommodate $t_i \geq 3$ people and $\sum t_i = mn$) for k different meals so that each person has every other person not in the same delegation for a neighbor exactly λ times?" In terms of graph theory, it is equivalent to the following general question.

Question 1.1. When does the graph $\lambda K(m:n)$ have a 2-factorization in which each 2-factor consists of cycles of lengths $t_1, t_2, \ldots, t_s$?

For $\lambda = 1$ and all tables have uniform sizes t, Alspach et al. [1], Hoffman et al. [5], and Liu [8] have obtained the following results which give the complete solution to the problem when all tables accommodate the same number t of people.

Theorem 1.2. (Alspach et al. [1] and Hoffman et al. [5]) *For $m = 1$ or 2, $K(m:n)$ has a C_t-factorization if and only if $m(n-1)$ is even and mn is divisible by t except for $t = 3$ and $K(m:n) = K(2:3), K(2:6)$.*

Theorem 1.3. (Liu [8]) *For $t \geq 3$ and $n \geq 2$, $K(m:n)$ has a C_t-factorization if and only if mn is divisible by t, $m(n-1)$ is even, t is even if $n = 2$, and $(m, n, t) \neq (2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)$.*

For $\lambda \geq 2$ and all tables have uniform sizes t, Ree [10] and Gvozdjak [4] obtained the following results.

Theorem 1.4. (Rees [10]) *For $\lambda \geq 1$, $\lambda K(m:n)$ has a C_3-factorization if and only if $\lambda m(n-1)$ is even, mn is divisible by 3, and $(m, n, \lambda) \notin \{(6, 3, 1), (2, 6, 1)\} \cup \{(2, 3, 2j+1), (1, 6, 4j+2) \mid j \geq 0\}$.*

Theorem 1.5. (Gvozdjak [4]) *For $t \geq 3$ and $n \geq 3$, $\lambda K(m:n)$ for $m = 1$ or 2 has a C_t-factorization if and only if $\lambda m(n-1)$ is even, mn is divisible by t, and $(m, n, t, \lambda) \notin \{(2, 6, 3, 1)\} \cup \{(2, 3, 3, 2j+1), (1, 6, 3, 4j+2) \mid j \geq 0\}$.*

Here, we will prove the following result which gives the complete solution to the problem for $\lambda \geq 2$ when all tables have uniform sizes t.

Theorem 1.6. *For $t \geq 3$, $n \geq 2$ and $\lambda \geq 2$, $\lambda K(m:n)$ has a C_t-factorization if and only if mn is divisible by t, $\lambda m(n-1)$ is even, t is even when $n = 2$, and $(m, n, t, \lambda) \notin \{(2, 3, 3, 2j+1), (1, 6, 3, 4j+2) \mid j \geq 0\}$.*

2. Proof of Theorem 1.6

We first recall that a $\{C_3, C_5\}$-factorization of a graph G is a 2-factorization such that each 2-factor is a disjoint union of C_3's and C_5's. Theorems 8 and 26 in [1] have characterized those K_n and $K(2:n)$ which have $\{C_3, C_5\}$-factorizations. Here, we will first show that for $q \geq 4$, $2K_{2q}$ has a $\{C_3, C_5\}$-factorization through a recursive construction.

Lemma 2.1. *$2K_8$ has a $\{C_3, C_5\}$-factorization.*

Proof. Let the vertex set of K_8 be $Z_7(=\{i \mid 0 \leq i \leq 6\}) \cup \{a\}$. Let F_0 be the following 2-factor for $2K_8$:

$$0, 1, 3, 6, 2, 0; \quad 4, 5, a, 4.$$

Then we claim that $\phi = \{F_k \mid 0 \leq k \leq 6\}$ is a $\{C_3, C_5\}$-factorization of $2K_8$, where F_k is obtained from F_0 by replacing each vertex i by $i + k \pmod 7$ while fixing the vertex a. A convenient way to see the construction is to place the vertices $0, 1, 2, \ldots, 6$ in 7 equally divided locations around a circle and place the vertex a in the center of the circle, and then rotate the starter 2-factor F_0 around the circle to obtain the $\{C_3, C_5\}$-factorization ϕ of $2K_8$. To check that each edge is used exactly twice after rotation, one need only to check that the starter F_0 satisfies the following conditions: (1) for $1 \leq d \leq 3$, exactly two edges of difference d (mod 7) are used; (2) exactly two edges adjacent to the vertex a are used. ∎

Lemma 2.2. *$2K_{14}$ has a $\{C_3, C_5\}$-factorization.*

Proof. Similarly to the proof of Lemma 2.1, let the vertex set of K_{14} be $Z_{13}(=\{i \mid 0 \leq i \leq 12\}) \cup \{a\}$. Let F_0 be the following 2-factor of $2K_{14}$:

$$0, 1, 7, 12, 2, 0; \quad 3, 5, 11, 3; \quad 6, 9, 10, 6; \quad 4, 8, a, 4.$$

Then we claim that $\phi = \{F_k \mid 0 \leq k \leq 12\}$ is a $\{C_3, C_5\}$-factorization of $2K_{14}$, where F_k is obtained from F_0 by replacing each vertex i by $i+k$(mod 13) while fixing the vertex a. ∎

Lemma 2.3. *$2K_{22}$ has a $\{C_3, C_5\}$-factorization.*

Proof. Similarly to the proof of Lemma 2.1, let the vertex set of K_{22} be $Z_{21}(=\{i \mid 0 \leq i \leq 20\}) \cup \{a\}$. Let F_0 be the following 2-factor of $2K_{22}$:

$$0, 1, 11, 20, 2, 0; \quad 3, 18, 4, 16, 19, 3;$$

$$5, 9, 13, 5; \quad 6, 12, 17, 6; \quad 7, 14, 15, 7; \quad 8, 10, a, 8.$$

Then we claim that $\phi = \{F_k \mid 0 \leq k \leq 20\}$ is a $\{C_3, C_5\}$-factorization of $2K_{22}$, where F_k is obtained from F_0 by replacing each vertex i by $i + k$ (mod 21) while fixing the vertex a. ∎

To introduce our recursive construction, we need the following concept of a cycle frame which was introduced in [1] and [11]. Let G be a complete multipartite graph and let J be a set of positive integers. A (G, J)-*cycle frame* is an edge decomposition of G, say $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$, such that

(1) every F_i is a 2-factor of $G-P$ for some partite set P of G,
(2) for every cycle $C \in F_i$, $1 \le i \le r$, $|C| \in J$.

When G is a complete equipartite graph $K(a:b)$, such a cycle frame is said to be of type a^b. From the definition, it follows that $r = \frac{|V(G)|}{2}$ and that for each partite set P of G, $|P|/2$ of the F_i's are 2-factors of $G-P$.

An easy example of a cycle frame of type 2^b with $J=\{3\}$ can be obtained from a Kirkman triple system $KTS(2b+1)$ by deleting a point and all triples containing that point. In general, Stinson has established the following necessary and sufficient condition for the existence of a cycle frame of type a^b with $J=\{3\}$ (see [11, Theorem 4.5]), where blocks of size 3 are also cycles of length 3.

Theorem 2.4. (Stinson [11]) *There exists a cycle frame of type a^u with cycles of length 3 if and only if a is even, $u \ge 4$, and $a(u-1) \equiv 0 \pmod 3$.*

Let J be a set of positive integers. In the following discussions, for convenience, a 2-factorization of $2K_n$ with cycle lengths in J is called an $(n,J)_2$-*resolvable cycle design* (denoted by $(n,J)_2$-RCD). An $(n,J)_2$-*RCD missing a sub* $(w,J)_2$-*RCD* is an edge decomposition $\Phi = \{Q_1,\dots,Q_{w-1}\} \cup \{F_1,F_2,\dots,F_{(n-w)}\}$ of $2K_n - E(2K_w)$ such that

(1) $Q_1, Q_2, \dots, Q_{w-1}$ are 2-factors of $2K_n - 2K_w$,
(2) $F_1, F_2, \dots, F_{n-w}$ are 2-factors of $2K_n$,
(3) every cycle in each of Q_i's and F_j's has length in J.

The next two lemmas give building blocks in the following recursive construction.

Lemma 2.5. *There is a $(8,\{3,5\})_2$-RCD missing a $(2,\{3,5\})_2$-RCD.*

Proof. Let the vertex set of K_8 be $Z_6(=\{i \mid 0 \le i \le 5\}) \cup \{a_1,a_2\}$. We define the desired edge decomposition of $2K_8 - E(2K_2) = 2K_8 - \{a_1a_2, a_1a_2\}$ to be $\Phi = \{Q_1, F_0, F_1, F_2, F_3, F_4, F_5\}$, where

$$Q_1 = 0,2,4,0; \quad 1,3,5,1,$$

$$F_0 = 0,1,4,0; \quad 2,3,a_1,5,a_2,2,$$

and for each $1 \le k \le 5$, F_k is obtained from F_0 by replacing each vertex i by $i+k$ (mod 6) while fixing the vertices a_1, a_2. Then it is easy to check that Φ is a $(8,\{3,5\})_2$-RCD missing a $(2,\{3,5\})_2$-RCD. ∎

Lemma 2.6. *There is a $(10,\{3,5\})_2$-RCD missing a $(4,\{3,5\})_2$-RCD.*

Proof. Let the vertex set of K_{10} be $Z_6(=\{i \mid 0 \le i \le 5\}) \cup \{a_1,a_2,a_3,a_4\}$ and let the subgraph K_4 of K_{10} have vertex set $\{a_1,a_2,a_3,a_4\}$. We define the desired edge decomposition of $2K_{10} - E(2K_4)$ to be $\Phi = \{Q_1,Q_2,Q_3,F_0,F_1,F_2,F_3,F_4,F_5\}$, where

$$Q_1 = 0,1,2,0; \quad 3,4,5,3,$$

$$Q_2 = 1,2,3,1; \quad 4,5,0,4,$$

$$Q_1 = 2,3,4,2; \quad 5,0,1,5,$$

$$F_0 = 1,a_1,2,5,a_2,1; \quad 0,a_3,3,a_4,4,0,$$

and for each $1 \le k \le 5$, F_k is obtained from F_0 by replacing each vertex i by $i+k$ (mod 6) while fixing the vertices a_1, a_2, a_3, a_4. Then it is easy to check that Φ is a $(10, \{3, 5\})_2$-RCD missing a $(4, \{3, 5\})_2$-RCD. ∎

Recall that a cycle frame of type a^b is based on the graph $K(a:b)$. Let the partite sets of $K(a:b)$ be $M_1, M_2, \ldots, M_b$. One can verify the following construction as follows: add a new set W of w vertices to $K(a:b)$, fill $M_i \cup W$ with an $(a+w, J)_2$-RCD missing a sub $(w, J)_2$-RCD for $1 \le i \le b-1$, and fill $M_b \cup W$ with an $(a+w, J)_2$-RCD.

Construction 2.7 (Filling in Holes). *Suppose there exists a (G, J)-cycle frame of type a^b and let $w \ge 0$. Suppose there exists an $(a+w, J)_2$-RCD missing a sub $(w, J)_2$-RCD. Also, suppose there exists an $(a+w, J)_2$-RCD. Then there exists an $(ab+w, J)_2$-RCD.*

Theorem 2.8. *For $q \ge 4$, $2K_{2q}$ has a $\{C_3, C_5\}$-factorization.*

Proof. For $q \equiv 0 \pmod 3$ (or mod 5), it follows from Theorem 1.5 that $2K_{2q} = 2K(1:2q)$ has a C_3-factorization (or a C_5-factorization, respectively), thus a $\{C_3, C_5\}$-factorization. Together with Lemmas 2.1–2.3, we have the result for $8 \le 2q \le 24$. Now, we assume $2q \ge 26$ and $q \not\equiv 0 \pmod 3$. By Theorem 2.4, there exists a cycle frame of type 6^u for $u \ge 4$. For $q = 3u+1 \ge 13$, we apply Construction 2.7 with $w = 2$ and Lemmas 2.1 and 2.5 to obtain a $(6u+2, \{3, 5\})_2$-RCD, i.e., a $\{C_3, C_5\}$-factorization of $2K_{2q}$. For $q = 3u+2 \ge 14$, since $2K_{10}$ has a C_5-factorization by Theorem 1.5 which gives a $(10, \{3, 5\})_2$-RCD, we apply Construction 2.7 with $w = 4$ and Lemma 2.6 to obtain a $(6u+4, \{3, 5\})_2$-RCD, i.e., a $\{C_3, C_5\}$-factorization of $2K_{2q}$. ∎

Next, we introduce a useful graph operation called lexicographic product. The *lexicographic product* $G = G_1[G_2]$ of two graphs G_1 and G_2 has vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, v_1)(u_2, v_2) \mid \text{either } u_1u_2 \in E(G_1) \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)\}$. For convenience, when G_2 is a graph on m vertices with no edge at all, $G_1[G_2]$ is simply denoted by $G_1[m]$. Consequently, $G[m]$ can be viewed as the graph with vertex set $V(G[m]) = \{(u_i, j) \mid u_i \in V(G) \text{ and } 1 \le j \le m\}$ and edge set $E(G[m]) = \{(u_i, a)(u_j, b) \mid u_iu_j \in E(G) \text{ and } 1 \le a, b \le m\}$. From the definition, it is easy to see that if $m = m_1m_2$, then $G[m] \cong G[m_1][m_2]$. In particular, we have the following remark which will be used implicitly very often throughout the paper.

Remark. For $m = m_1m_2$, the complete equipartite graph $K(m:n)$ satisfies

$$K(m:n) \cong K_n[m] \cong K(m_1:n)[m_2] \cong K_n[m_1][m_2].$$

We will write $K(m:n) = K_n[m] = K(m_1:n)[m_2] = K_n[m_1][m_2]$.

The following results involving lexicographic product are very useful in our discussion.

Theorem 2.9. (Baranyi and Szasz [2]) *The lexicographic product of two Hamiltonian decomposable graphs is Hamiltonian decomposable.*

Corollary 2.10. *$C_p[r]$ has a C_{pr}-factorization.*

Lemma 2.11. (Alsopach et al. [1]) *Let s be an odd integer and p be a prime so that $3 \le s \le p$. Then $C_s[p]$ has a C_p-factorization.*

The next lemma is a special case with $s = e = a_1 = a_2 = \cdots = a_k$ of [9, Corollary 5.7].

Lemma 2.12. (Piotrowski [9]) *For $s \geq 4$, $C_s[p]$ has a C_s-factorization except for $p = 2$ and s odd.*

Lemma 2.13. *For any odd integer $p \geq 5$, $2K(p:4)$ has a C_p-factorization.*

Proof. Let the 4 partite sets for $K(p:4)$ be B_0, B_1, B_2 and $A = \{a_1, a_2, \ldots, a_p\}$, where $B_j = \{3i + j \mid 0 \leq i \leq p-1\}$ for $j = 0, 1, 2$. Then the differences (mod $3p$) of the edges of $K(p:4) - A$ form the set $S = \{1, 2, 3, \ldots, (3p-1)/2\} - \{3, 6, 9, \ldots, 3(p-1)/2\}$. For convenience, we place the vertices $1, 2, \ldots, 3p-1$ at $3p$ equally divided locations around a circle and place the vertices $a_1, a_2, \ldots, a_p$ in a row below the circle. Suppose that $2K(p:4)$ has a 2-factor F_0 consisting of disjoint C_p's such that (1) F_0 uses exactly two edges of difference d (mod $3p$) for each $d \in S$ and (2) F_0 uses exactly two edges adjacent to a_i for each a_i. Then we obtain a C_p-factorization $\phi = \{F_i \mid 0 \leq i \leq 3p-1\}$ of $2K(p:4)$ by rotating F_0 around the circle, i.e., each F_k is obtained from F_0 by replacing each vertex i by $i+k$ (mod $3p$) while fixing the vertices a_x. Thus, for the rest of the proof, we need only to find such a 2-factor F_0 satisfying the conditions (1) and (2).

We claim that the graph $2K(p:4) - A$ has a subgraph $H = P_{p-1}(1) \cup P_{p-1}(2) \cup P_4 \cup P_2$ which uses exactly two edges of difference d (mod $3p$) for each $d \in S$, where each $P_{p-1}(i)$ is a path on $p-1$ vertices and all the paths are vertex-disjoint. In fact, let

$$P_{p-1}(1) = 0, 3p-1, 1, 3p-3, 2, 3p-5, 3, 3p-7, \ldots, i, 3p-2i-1,$$

$$i+1, 3p-2(i+1)-1, \ldots, \frac{p-3}{2}, 3p-2\frac{p-3}{2}-1;$$

$$P_{p-1}(2) = p+3, p+4, p+2, p+6, \ldots, p+3-i, p+4+2i, \ldots,$$

$$p+3-\frac{p-3}{2}, p+4+2\frac{p-3}{2};$$

$$P_4 = \frac{p-1}{2}, 2p, \frac{p+1}{2}, 2p-2;$$

$$P_2 = \frac{p+5}{2}, 2p+5.$$

Then it is easy to check that these four paths are disjoint, each of $P_{p-1}(1)$ and $P_{p-1}(2)$ uses exactly one edge of difference d (mod $3p$) for each $d \in S - \{(3p-5)/2, (3p-1)/2\}$, the differences (mod $3p$) on the edges of P_4 are $(3p-1)/2, (3p-1)/2, (3p-5)/2$, and the edge of P_2 has the difference $(3p-5)/2$ (mod $3p$). Thus $H = P_{p-1}(1) \cup P_{p-1}(2) \cup P_4 \cup P_2$ uses exactly two edges of difference d (mod $3p$) for each $d \in S$ as desired.

Let W be the set of vertices in $K(p:4) - A$ which are not contained in H. Then W has $p-4$ vertices. Now, we form a 2-factor F_0 as follows: use a_1 to join the two end vertices of $P_{p-1}(1)$ to form a C_p and use a_2 to join the two end vertices of $P_{p-1}(2)$ to form the second C_p, use $(p-5)/2$ vertices in W and $(p-3)/2$ vertices a_i for $3 \leq i \leq (p+1)/2$ together with P_4 to form the third C_p, and use the remaining $(p-3)/2$

vertices in W and the remaining $(p-1)/2$ vertices a_i in A together with P_2 to form the fourth C_p. Then it is easy to see that F_0 is a 2-factor of $2K(p:4)$ consisting of disjoint C_p's which satisfies the conditions (1) and (2). This completes the proof. ∎

Lemma 2.14. *For any odd integer $p \geq 5$, $2K(p:6)$ has a C_p-factorization.*

Proof. Let the 6 partite sets for $K(p:6)$ be B_0, B_1, B_2, B_3, B_4, and $A = \{a_1, a_2, \ldots, a_p\}$, where $B_j = \{5i + j \mid 0 \leq i \leq p-1\}$ for $0 \leq j \leq 4$. Then the differences (mod $5p$) of the edges of $K(p:6) - A$ form the set $S = \{1, 2, 3, \ldots, (5p-1)/2\} - \{5, 10, 15, \ldots, 5(p-1)/2\}$. Similarly to the proof of Lemma 2.13, for convenience, we place the vertices $1, 2, \ldots, 5p-1$ at $5p$ equally divided locations around a circle and place the vertices $a_1, a_2, \ldots, a_p$ in a row below the circle. Suppose that $2K(p:6)$ has a 2-factor F_0 consisting of disjoint C_p's such that (1) F_0 uses exactly two edges of difference d (mod $5p$) for each $d \in S$ and (2) F_0 uses exactly two edges adjacent to a_i for each a_i. Then we obtain a C_p-factorization $\phi = \{F_i \mid 0 \leq i \leq 5p-1\}$ of $2K(p:6)$ by rotating F_0 around the circle, i.e., each F_k is obtained from F_0 by replacing each vertex i by $i+k$ (mod $5p$) while fixing the vertices a_x. Thus, for the rest of the proof, we need only to find such a 2-factor F_0 satisfying the conditions (1) and (2).

We claim that the graph $2K(p:6) - A$ has a subgraph $H = P_{p-1}(1) \cup P_{p-1}(2) \cup P_{p-1}(3) \cup P_{p-1}(4) \cup P_8 \cup P_2$ which uses exactly two edges of difference d (mod $5p$) for each $d \in S$, where each $P_{p-1}(i)$ is a path on $p-1$ vertices and the six paths are vertex-disjoint. In fact, let P be the following path on $2(p-1)$ vertices:

$$P = 1, 5p-1, 5p-2, 2, 5p-4, 3, 5p-5, 4, 5p-7, 5, 5p-8, \ldots, 2i, 5p-3i-1,$$

$$2i+1, 5p-3i-2, \ldots, p-3, 5p - \frac{3(p-3)}{2} - 1, p-2, 5p - \frac{3(p-3)}{2} - 2, p-1.$$

Then P uses exactly one edge of difference d (mod $5p$) for each $d \in S - \{3, (5p-3)/2, (5p-1)/2\}$. Since P has $2(p-1)$ vertices, we can separate P into two paths P' and P'', each having $p-1$ vertices, by removing the middle edge e from P, where $e = \frac{p-1}{2}, 5p - \frac{3(p-1)}{4} - 1$ for $p \equiv 1$ (mod 4) (i.e., $\frac{p-1}{2}$ even) and $e = \frac{p-1}{2}, 5p - \frac{3(p-3)}{4} - 2$ for $p \equiv 3$ (mod 4) (i.e., $\frac{p-1}{2}$ odd). Clearly, the edge e has difference b, where $b = \frac{5(p-1)}{4} + 1$ for $p \equiv 1$ (mod 4) and $b = \frac{5(p-3)}{4} + 3$ for $p \equiv 3$ (mod 4). Thus, $P' \cup P''$ uses exactly one edge of difference d (mod $5p$) for each $d \in S - \{3, b, (5p-3)/2, (5p-1)/2\}$. Furthermore, if necessary, we can modify these two subpaths P' and P'' so that the missing difference b can be replaced by $b+1$ for $p \equiv 1$ (mod 4) or $b-2$ for $p \equiv 3$ (mod 4). Suppose P' is the first half subpath, namely, from 1 to $\frac{p-1}{2}$. Then we can do the modification, if necessary, as follows: for $p \equiv 1$ (mod 4), we replace the edge $\frac{p+1}{2}, 5p - \frac{3(p-1)}{4} - 1$ on P'' by the edge $\frac{p+1}{2}, 5p - \frac{3(p-1)}{4}$; for $p \equiv 3$ (mod 4), we replace the edge $\frac{p+1}{2}, 5p - \frac{3(p-3)}{4} - 2$ on P'' by $\frac{p+1}{2}, 5p - \frac{3(p-3)}{4}$ and replace the subpath $\frac{p-3}{2}, 5p - \frac{3(p-3)}{4} - 1, \frac{p-1}{2}$ of P' by $\frac{p-3}{2}, 5p - \frac{3(p-3)}{4} - 3, \frac{p-1}{2}$. Let $P_{p-1}(1) = P'$ and $P_{p-1}(2) = P''$. Note that all $2(p-1)$ vertices of $P_{p-1}(1) \cup P_{p-1}(2)$ are on half of the circle, by using mirror image with respect to the line L through the point $5p - \frac{3(p-1)}{2} - 1$ and the middle of the two points p and $p+1$ on the circle, we obtain another two subpaths $P_{p-1}(3)$ and $P_{p-1}(4)$. It follows that $P_{p-1}(1) \cup P_{p-1}(2) \cup P_{p-1}(3) \cup P_{p-1}(4)$ uses exactly two edges of difference d (mod $5p$) for each $d \in S - \{3, y, (5p-3)/2, (5p-1)/2\}$, where $y = b$ or

$b+1$ for $p \equiv 1 \pmod 4$, $y = b$ or $b-2$ for $p \equiv 3 \pmod 4$. Next we find desired P_8 and P_2. Let $P^* = 5p - \frac{3(p-1)}{2}$, p, $5p - \frac{3(p-1)}{2} - 1$, $p+1$, $5p - \frac{3(p-1)}{2} - 2$. Then P^* uses exactly two edges of difference $d \pmod{5p}$ for each $d \in \{(5p-3)/2, (5p-1)/2\}$ and $H' = P^* \cup P_{p-1}(1) \cup P_{p-1}(2) \cup P_{p-1}(3) \cup P_{p-1}(4)$ is symmetric under the mirror image with respect to the line L and uses exactly two edges of difference $d \pmod{5p}$ for each $d \in S - \{3, y\}$. Note that the vertices $5p - 3i$ for $i = 0, 1, 2, \ldots, \frac{(p-3)}{2}$ are not contained in H' except possibly for those involved in the modifications mentioned above. We can find a proper choice of one end vertex a of P^* and a proper choice of y so that $a + y$ is not contained in H', where $a = 5p - \frac{3(p-1)}{2}$ or $5p - \frac{3(p-1)}{2} - 2$. Now attach $a+y$ to P^* through the edge $a, a+y$ and attach the mirror image of $a, a+y$ with respect to the line L to the other end of P^* to form a path P_7. We then attach a vertex not in H' to one end of P_7 through an edge with difference 3 to obtain P_8. Finally let P_2 be any edge between two unused vertices whose difference is 3. Thus, we have found $H = P_{p-1}(1) \cup P_{p-1}(2) \cup P_{p-1}(3) \cup P_{p-1}(4) \cup P_8 \cup P_2$ which uses exactly two edges of difference $d \pmod{5p}$ for each $d \in S$.

Let W be the set of vertices in $K(p:6) - A$ which are not contained in H. Then W has $p-6$ vertices. Now, for $p \geq 9$, we form a 2-factor F_0 as follows: use a_i to join the two end vertices of $P_{p-1}i$ to form a C_p for each $1 \leq i \leq 4$, use $(p-9)/2$ vertices in W and $(p-7)/2$ vertices a_i for $5 \leq i \leq (p+1)/2$ together with P_8 to form the fifth C_p, and use the remaining $(p-3)/2$ vertices in W and the remaining $(p-1)/2$ vertices a_i in A together with P_2 to form the sixth C_p. Then it is easy to see that F_0 is a 2-factor of $2K(p:6)$ consisting of disjoint C_p's which satisfies the conditions (1) and (2). For $p = 5$ or 7, one can find such a 2-factor F_0 in the same manner. This completes the proof. ∎

Theorem 2.15. *For any integer $q \geq 2$ and odd integer $p \geq 3$, $2K(p:2q)$ has a C_p-factorization.*

Proof. By Theorem 1.4, we may assume $p \geq 5$. By Lemmas 2.13 and 2.14, we may assume $q \geq 4$. It follows from Theorem 2.8 that $2K_{2q}$ has a $\{C_3, C_5\}$-factorization which implies that $2K(p:2q) = 2K_{2q}[p]$ has a $\{C_3[p], C_5[p]\}$-factorization. Since each of $C_3[p]$ and $C_5[p]$ has a C_p-factorization for odd $p \geq 5$ by Lemma 2.11, $2K(p:2q)$ has a C_p-factorization. ∎

Lemma 2.16. *For any odd integer $p \geq 3$, $2K(p:2) = 2K(p,p)$ has a Hamiltonian decomposition, i.e., a C_{2p}-factorization.*

Proof. Let $A = \{a_i \mid 0 \leq i \leq p-1\}$ and $B = \{b_i \mid 0 \leq i \leq p-1\}$ be the two partite sets of $K(p,p)$. Let $F_j = \{(a_i, b_{i+j}) \mid 0 \leq i \leq p-1\}$ for $0 \leq j \leq p-1$ with the subscripts taken modulo p. Then it is easy to check that $F_j \cup F_{j+1}$, for $0 \leq j \leq p-1$, form a Hamiltonian decomposition of $2K(p,p)$, where the subscripts are taken modulo p. ∎

We are now ready to prove our main result Theorem 1.6.

Proof of Theorem 1.6. The necessity is clear because we are dealing with 2-factorizations of $\lambda K(m:n)$ in which each 2-factor is disjoint union of cycles C_ts.

For the sufficiency, by Theorems 1.3, 1.4 and 1.5, we need only to consider the case when $\lambda = 2$, $m(n-1)$ is odd, $t \geq 4$, and $m \geq 3$. This implies that $m \geq 3$ is odd and $n \geq 2$

is even. Since $t|mn$, we let $t = t_1t_2$ with $t_1|m$, $t_2|n$, and $\gcd(t_1,n) = 1$. Then t_1 is odd. If $t_2 \geq 3$, then it follows from Theorem 1.5 that $2K_n = 2K(1:n)$ has a C_{t_2}-factorization which implies that $2K(t_1:n) = 2K_n[t_1]$ has a $C_{t_2}[t_1]$-factorization, thus a C_t-factorization by Corollary 2.10. Then $2K(m:n) = 2K(t_1:n)[m/t_1]$ has a $C_t[m/t_1]$-factorization, and so a C_t-factorization by Lemma 2.12. Therefore, we assume $t_2 = 1$ or 2. We consider the following two cases.

Case 1. $t = t_1t_2$ is even. Then t_2 is even as t_1 is odd, and so $t_2 = 2$. We first assume $n = 2$. By Lemma 2.16, $2K(t_1:2)$ has a C_t-factorization which implies that $2K(m:2) = 2K(t_1:2)[m/t_1]$ has a $C_t[m/t_1]$-factorization, thus a C_t-factorization by Lemma 2.12. Now, we assume $n \geq 4$. Note that $2K(m:n)$ is a disjoint union of two factors $F = \cup 2K(m:2)$ and $2K(2m:n/2)$. Since $2K(m:2)$ has a C_t-factorization by the previous argument and $K(2m:n/2)$ has a C_t-factorization by Theorem 1.3, $2K(m:n)$ has a C_t-factorization.

Case 2. $t = t_1t_2$ is odd. This implies that $n \geq 3$ and both t_1 and t_2 are odd. Thus, we must have $t_2 = 1$ and $t = t_1|m$. Since $n \geq 4$ is even, it follows from Theorem 2.15 that $2K(t:n)$ has a C_t-factorization which implies that $2K(m:n) = 2K(t:n)[m/t]$ has a $C_t[m/t]$-factorization, thus a C_t-factorization by Lemma 2.12. ∎

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