Surveys in Geometric Analysis 2017

(几何分析综述2017)

Editor-in-Chief: Gang Tian(田刚)

Edited by Qing Han(韩青) & Zhenlei Zhang(张振雷)

SCIENCE PRESS
BEIJING

Positive Scalar Curvature and Connected Sums

GUANGXIANG SU

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China

E-mail: guangxiangsu@nankai.edu.cn

WEIPING ZHANG

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China

E-mail: weiping@nankai.edu.cn

Abstract Let N be a closed enlargeable manifold in the sense of Gromov-Lawson and M a closed spin manifold of equal dimension, a famous theorem of Gromov-Lawson states that the connected sum $M\#N$ admits no metric of positive scalar curvature. We present a potential generalization of this result to the case where M is nonspin. We use index theory for Dirac operators to prove our result.

0. Introduction

It has been an important subject in differential geometry to study when a smooth manifold carries a Riemannian metric of positive scalar curvature (cf. [6, Chap. IV]). A famous theorem of Gromov and Lawson [4], [5] states that an enlargeable manifold (in the sense of [5]) does not carry a metric of positive scalar curvature.

Definition 0.1. (Gromov-Lawson [5, Definition 5.5]) *One calls a closed manifold N (carrying a metric g^{TN}) an enlargeable manifold if for any $\epsilon > 0$, there is a covering manifold $\widehat{N}_\epsilon \to N$, with $\widehat{N}_\epsilon$ being spin, and a smooth map $f : \widehat{N}_\epsilon \to S^{\dim N}(1)$ (the standard unit sphere), which is constant near infinity and has non-zero degree, such that for any $X \in \Gamma(T\widehat{N}_\epsilon)$, $|f_*(X)| \leqslant \epsilon|X|$.*

It is clear that the enlargeability does not depend on the metric g^{TN}.

We assume from now on that N is a closed enlargeable manifold. Let M be a closed manifold such that there is a closed codimension two submanifold $W \subset M$ such that $M \setminus W$ is spin. Without loss of generality, we assume that $\dim M = \dim N = n$ is even. Let h^{TN} be a metric on TN.

We fix a point $p \in N$. For any $r \geqslant 0$, let $B_p^N(r) = \{y \in N : d(p, y) \leqslant r\}$. Let $a_0 > 0$ be a fixed sufficiently small number. Then the connected sum $M \# N$ can be constructed so that the hypersurface $\partial B_p^N(a_0)$, which is the boundary of $B_p^N(a_0)$, cuts $M \# N$ into two parts: the part $N \setminus B_p^N(a_0)$ and the rest part coming from M (by attaching the boundary of a ball in $M \setminus W$ to $\partial B_p^N(a_0)$).

Let $\varphi : M \# N \to [0, 1]$ be an arbitrary smooth function such that $\varphi \equiv 1$ on $N \setminus B_p^N(a_0)$ and $\mathrm{Supp}(\varphi) \subseteq M \# N \setminus W$. The main result of this paper can be stated as follows.

Theorem 0.2. *There is no metric $g^{T(M\#N)}$ on $T(M\#N)$ such that the associated scalar curvature $k^{T(M\#N)}$ verifies the following inequality on* $\mathrm{Supp}(\varphi)$,

$$c - 6|d\varphi|^2_{g^{T(M\#N)}} \geqslant \max\left\{0, \frac{3c}{2} - \frac{k^{T(M\#N)}}{4}\right\} \tag{0.1}$$

for some constant $c > 0$.

When M is spin, one can take $W = \emptyset$, $\varphi \equiv 1$ on $M \# N$ and $c > 0$ small enough to recover the theorem of Gromov-Lawson [4], [5] mentioned at the begining.

Our proof of Theorem 0.2 is index theoretic and is inspired by [8], where a new proof of the above mentioned Gromov-Lawson theorem is given without using index theorems on noncompact manifolds. The details will be carried out in Section 1.

1. Proof of Theorem 0.2

Assume there is a metric $g^{T(M\#N)}$ on $T(M\#N)$ such that (0.1) holds for $c = \alpha^2 > 0$.

For any $\epsilon > 0$, let $\pi : \widehat{N}_\epsilon \to N$ be a covering manifold verifying Definition 0.1, carrying lifted geometric data from that of N. Let $a_0 > 0$ be small enough so that for any p', $q' \in \pi^{-1}(p)$ with $p' \neq q'$, $B_{p'}^{\widehat{N}_\epsilon}(4a_0) \cap B_{q'}^{\widehat{N}_\epsilon}(4a_0) = \emptyset$.[1] It is clear that one can choose $a_0 > 0$ not depending on ϵ.

[1] Here and in what follows, the involved balls are determined by h^{TN}.

Let $h : N \to N$ be a smooth map such that $h = \mathrm{Id}$ on $N \setminus B_p^N(3a_0)$, while $h(B_p^N(2a_0)) = \{p\}$. It lifts to a map $\widehat{h} : \widehat{N}_\epsilon \to \widehat{N}_\epsilon$ verifying that $\widehat{h} = \mathrm{Id}$ on $\widehat{N}_\epsilon \setminus \bigcup_{p' \in \pi^{-1}(p)} B_{p'}^{\widehat{N}_\epsilon}(3a_0)$, while for any $p' \in \pi^{-1}(p)$, $\widehat{h}(B_{p'}^{\widehat{N}_\epsilon}(2a_0)) = \{p'\}$.

Let $f : \widehat{N}_\epsilon \to S^n(1)$ be as in Definition 0.1. Set $\widehat{f} = f \circ \widehat{h} : \widehat{N}_\epsilon \to S^n(1)$. Then $\deg(\widehat{f}) = \deg(f)$ and there is a constant $c' > 0$ such that for any $X \in \Gamma(T\widehat{N}_\epsilon)$, one has

$$\left|\widehat{f}_*(X)\right| \leqslant c' \, \epsilon \, |X|. \tag{1.1}$$

To simplify the presentation, we assume first that each $\widehat{N}_\epsilon$ is compact, i.e., N is a *compactly* enlargeable manifold.

Since $M \setminus W$ is spin, one can construct a compact spin manifold with boundary $M_W \subset M \setminus W$ such that $\partial M_W \subseteq M \setminus \mathrm{Supp}(\varphi)$. Let M'_W be another copy of M_W. Then one gets a closed spin manifold by gluing M_W and M'_W along the boundary. We denote the resulting double by $\widetilde{M}_W$. Then one can extend the connected sum $M_W \# N$ to $\widetilde{M}_W \# N$ obviously. It lifts natually to $\widehat{N}_\epsilon$ where near each $p' \in \pi^{-1}(p)$, we do the lifted connected sum. We denote the resulting manifold by $\widehat{M_W \# \widehat{N}_\epsilon}$. Clearly, the metric $g^{T(M\#N)}$ induces a metric $g^{T(\widetilde{M}_W\#N)}$ such that $g^{T(\widetilde{M}_W\#N)}|_{M_W\#N} = g^{T(M\#N)}|_{M_W\#N}$. Let $k^{T(\widetilde{M}_W\#N)}$ be the associated scalar curvature. They determine the corresponding metric $g^{T(\widehat{M}_W\#\widehat{N}_\epsilon)}$ and scalar curvature $k^{T(\widehat{M}_W\#\widehat{N}_\epsilon)}$ on $\widehat{M_W\#\widehat{N}_\epsilon}$.

The cut-off function φ extends to $\widetilde{M}_W\#N$ by setting $\varphi(M'_W) = 0$. It lifts to $\widehat{M_W\#\widehat{N}_\epsilon}$ obviously and we still denote the lifting by φ.

We extend $\widehat{f} : \widehat{N}_\epsilon \to S^n(1)$ to $\widehat{f} : \widehat{M_W\#\widehat{N}_\epsilon} \to S^n(1)$ by setting $\widehat{f}(\widetilde{M}_W\#B_{p'}(4a_0)) = f(p')$ for any $p' \in \pi^{-1}(p)$.

Following [4], [5], let (E_0, g^{E_0}) be a Hermitian vector bundle on $S^n(1)$ carrying a Hermitian connection ∇^{E_0} such that

$$\langle \mathrm{ch}\,(E_0)\,, [S^n(1)] \rangle \neq 0. \tag{1.2}$$

Let $(E_1 = \mathbf{C}^k|_{S^n(1)}, g^{E_1}, \nabla^{E_1})$, with $k = \mathrm{rk}(E_0)$, be the canonical Hermitian trivial vector bundle on $S^n(1)$.

For any $p' \in \pi^{-1}(p)$, let $v_{f(p')} : \Gamma(E_0|_{f(p')}) \to \Gamma(E_1|_{f(p')})$ be an isometry. Let $v^*_{f(p')} : \Gamma(E_1|_{f(p')}) \to \Gamma(E_0|_{f(p')})$ be the adjoint of $v_{f(p')}$ with respect to $g^{E_0}|_{f(p')}$ and $g^{E_1}|_{f(p')}$. Set

$$V_{f(p')} = v_{f(p')} + v^*_{f(p')}. \tag{1.3}$$

Let $(\xi, g^{\xi}, \nabla^{\xi}) = (\xi_0 \oplus \xi_1, g^{\xi_0} \oplus g^{\xi_1}, \nabla^{\xi_0} \oplus \nabla^{\xi_1}) = (\widehat{f}^* E_0 \oplus \widehat{f}^* E_1, \widehat{f}^* g^{E_0} \oplus \widehat{f}^* g^{E_1}, \widehat{f}^* \nabla^{E_0} \oplus \widehat{f}^* \nabla^{E_1})$ be the $\mathbf{Z}_2$-graded Hermitian vector bundle with Hermitian connection over $\widehat{M}_W \# \widehat{N}_\epsilon$. Let $R^{\xi} = (\nabla^{\xi})^2$ be the curvature of ∇^{ξ}.

Let $D^{\xi} : \Gamma(S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi) \to \Gamma(S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi)$ be the canonical (twisted) Dirac operator (cf. [6]) associated to $(T(\widehat{M}_W \# \widehat{N}_\epsilon), g^{T(\widehat{M}_W \# \widehat{N}_\epsilon)})$ and $(\xi, g^{\xi}, \nabla^{\xi})$. Let $D^{\xi}_{\pm} : \Gamma((S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi)_{\pm}) \to \Gamma((S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi)_{\mp})$ be the obvious restrictions, where $(S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi)_+ = S_+(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \otimes \xi_0 \oplus S_-(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \otimes \xi_1$, while $(S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi)_- = S_-(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \otimes \xi_0 \oplus S_+(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \otimes \xi_1$. By the Atiyah-Singer index theorem [1] (cf. [6]) and [5], one has

$$\begin{aligned} \operatorname{ind}\left(D^{\xi}_{+}\right) &= \left\langle \widehat{A}\left(T\left(\widehat{M}_W \# \widehat{N}_\epsilon\right)\right)\left(\operatorname{ch}(\xi_0) - \operatorname{ch}(\xi_1)\right), \left[\widehat{M}_W \# \widehat{N}_\epsilon\right]\right\rangle \\ &= (\deg(f)) \left\langle \operatorname{ch}(E_0), [S^n(1)]\right\rangle. \end{aligned} \tag{1.4}$$

Following [2, p. 115], let $\varphi_1, \varphi_2 : \widehat{M}_W \# \widehat{N}_\epsilon \to [0,1]$ be defined by

$$\varphi_1 = \frac{\varphi}{(\varphi^2 + (1-\varphi)^2)^{\frac{1}{2}}}, \quad \varphi_2 = \frac{1-\varphi}{(\varphi^2 + (1-\varphi)^2)^{\frac{1}{2}}}. \tag{1.5}$$

Then $\varphi_1^2 + \varphi_2^2 = 1$.

Recall that $\alpha^2 = c > 0$. Let $D^{\xi}_{\alpha} : \Gamma(S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi) \to \Gamma(S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi)$ be the deformed operator defined by

$$D^{\xi}_{\alpha} = D^{\xi} + \alpha \sum_{p' \in \pi^{-1}(p)} \varphi_2 V_{p'}, \tag{1.6}$$

where $V_{p'} = \widehat{f}^{-1}(V_{f(p')})$ lives on the lift of $M_W \# B_p(4a_0)$ near p'.

From (1.6), it is easy to verify that

$$\left(D^{\xi}_{\alpha}\right)^2 = \left(D^{\xi}\right)^2 + \alpha \sum_{p' \in \pi^{-1}(p)} c(d\varphi_2) V_{p'} + \alpha^2 \varphi_2^2. \tag{1.7}$$

Thus for any $s \in \Gamma(S(T(\widehat{M}_W \# \widehat{N}_\epsilon)) \widehat{\otimes} \xi)$, we have

$$\begin{aligned} \left\|D^{\xi}_{\alpha} s\right\|^2 &= \|D^{\xi} s\|^2 + \alpha \sum_{p' \in \pi^{-1}(p)} \langle c(d\varphi_2) V_{p'} s, s\rangle + \alpha^2 \|\varphi_2 s\|^2 \\ &= \left\|\varphi_1 D^{\xi} s\right\|^2 + \left\|\varphi_2 D^{\xi} s\right\|^2 + \alpha \sum_{p' \in \pi^{-1}(p)} \langle c(d\varphi_2) V_{p'} s, s\rangle + \alpha^2 \|\varphi_2 s\|^2. \end{aligned} \tag{1.8}$$

By direct computation we have for $i = 1, 2$ that

$$\left\|\varphi_i D^{\xi} s\right\|^2 = \left\|D^{\xi}\left(\varphi_i s\right)\right\|^2 - \left\||d\varphi_i|\, s\right\|^2 - \left\langle D^{\xi} s, \frac{c\left(d\varphi_i^2\right)}{2} s\right\rangle - \left\langle \frac{c\left(d\varphi_i^2\right)}{2} s, D^{\xi} s\right\rangle. \tag{1.9}$$

By (1.8) and (1.9), we have

$$\begin{aligned}\left\|D_{\alpha}^{\xi} s\right\|^2 = & \sum_{i=1}^{2}\left(\left\|D^{\xi}\left(\varphi_i s\right)\right\|^2 - \left\||d\varphi_i|\, s\right\|^2\right)\\ & + \alpha \sum_{p' \in \pi^{-1}(p)} \left\langle c\left(d\varphi_2\right) V_{p'} s, s\right\rangle + \alpha^2 \left\|\varphi_2 s\right\|^2 .\end{aligned} \tag{1.10}$$

By (1.5), we have

$$d\varphi_1 = \frac{\varphi_2 d\varphi}{\varphi^2 + (1-\varphi)^2}, \quad d\varphi_2 = -\frac{\varphi_1 d\varphi}{\varphi^2 + (1-\varphi)^2}. \tag{1.11}$$

Let $e_1, \cdots, e_n$ be an orthonormal basis of $(T(\widehat{M}_W \# \widehat{N}_\epsilon), g^{T(\widehat{M}_W \# \widehat{N}_\epsilon)})$. By (1.1), (1.10), (1.11), the Lichnerowicz formula [7] (cf. [6]) and proceed as in [5], one has

$$\begin{aligned}
&\left\|D_{\alpha}^{\xi} s\right\|^2 \\
=& -\left\langle \Delta\left(\varphi_1 s\right), \varphi_1 s\right\rangle + \left\langle \frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} \varphi_1 s, \varphi_1 s\right\rangle + \left\|D^{\xi}\left(\varphi_2 s\right)\right\|^2 \\
& - \left\|\frac{|d\varphi|}{\varphi^2 + (1-\varphi)^2} s\right\|^2 - \alpha \sum_{p' \in \pi^{-1}(p)} \left\langle \frac{\varphi_1 c(d\varphi) V_{p'}}{\varphi^2 + (1-\varphi)^2} s, s\right\rangle + \alpha^2 \left\|\varphi_2 s\right\|^2 \\
& + \frac{1}{2} \sum_{i,j=1}^{n} \left\langle c\left(e_i\right) c\left(e_j\right) R^{\xi}\left(e_i, e_j\right) s, s\right\rangle \\
\geqslant & \left\langle \left(\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} \varphi_1^2 + \alpha^2 \varphi_2^2 - \frac{|d\varphi|^2}{\left(\varphi^2 + (1-\varphi)^2\right)^2} - \alpha \sum_{p' \in \pi^{-1}(p)} \frac{\varphi_1 c(d\varphi) V_{p'}}{\varphi^2 + (1-\varphi)^2} \right) s, s\right\rangle \\
& + \frac{1}{2} \sum_{i,j=1}^{n} \left\langle c\left(e_i\right) c\left(e_j\right) \widehat{f}^* \left(\left(\nabla^{E_0}\right)^2 \left(\widehat{f}_*\left(e_i\right), \widehat{f}_*\left(e_j\right)\right)\right) s, s\right\rangle \\
\geqslant & \left\langle \left(\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} \varphi_1^2 + \alpha^2 \varphi_2^2 - \frac{\alpha^2 \varphi_1^2}{2}\bigg|_{\mathrm{Supp}(d\varphi)} - \frac{3}{2} \frac{|d\varphi|^2}{\left(\varphi^2 + (1-\varphi)^2\right)^2} \right) s, s\right\rangle \\
& + \left\langle O\left(\epsilon^2\right) s, s\right\rangle_{\widehat{N}_\epsilon \setminus \cup_{p' \in \pi^{-1}(p)} B_{p'}^{\widehat{N}_\epsilon}(a_0)} \\
\geqslant & \left\langle \left(\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} \varphi_1^2 + \alpha^2 \varphi_2^2 - \frac{\alpha^2 \varphi_1^2}{2}\bigg|_{\mathrm{Supp}(d\varphi)} - 6|d\varphi|^2 \right) s, s\right\rangle \\
& + \left\langle O\left(\epsilon^2\right) s, s\right\rangle_{\widehat{N}_\epsilon \setminus \cup_{p' \in \pi^{-1}(p)} B_{p'}^{\widehat{N}_\epsilon}(a_0)} .
\end{aligned} \tag{1.12}$$

For any $x \in \mathrm{Supp}(d\varphi)$, if $\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} - \frac{3\alpha^2}{2} \geqslant 0$ at x, then one has at x that, in view of (0.1),

$$\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4}\varphi_1^2 + \alpha^2\varphi_2^2 - \frac{\alpha^2\varphi_1^2}{2}\bigg|_{\mathrm{Supp}(d\varphi)} - 6|d\varphi|^2$$
$$= \left(\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} - \frac{3\alpha^2}{2}\right)\varphi_1^2 + \alpha^2 - 6|d\varphi|^2 \geqslant 0, \tag{1.13}$$

while if $\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} - \frac{3\alpha^2}{2} \leqslant 0$ at x, then by (0.1), one has at x that

$$\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4}\varphi_1^2 + \alpha^2\varphi_2^2 - \frac{\alpha^2\varphi_1^2}{2}\bigg|_{\mathrm{Supp}(d\varphi)} - 6|d\varphi|^2$$
$$= \left(\frac{3\alpha^2}{2} - \frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4}\right)\varphi_2^2 + \frac{k^{T(\widetilde{M}_W \# N)}}{4} - \frac{\alpha^2}{2} - 6|d\varphi|^2 \geqslant 0. \tag{1.14}$$

On the other hand, (0.1) implies that

$$\frac{k^{T(\widehat{M}_W \# \widehat{N}_\epsilon)}}{4} \geqslant \frac{\alpha^2}{2} \tag{1.15}$$

on $\pi^{-1}(N \setminus B_p(a_0)) = \widehat{N}_\epsilon \setminus \cup_{p' \in \pi^{-1}(p)} B_{p'}^{\widehat{N}_\epsilon}(a_0)$, on which $\varphi \equiv 1$.

From (1.12)-(1.15), we see that if (0.1) holds for $c = \alpha^2$, then one has

$$\left\|D_\alpha^\xi s\right\|^2 \geqslant \left\langle \left(\frac{\alpha^2}{2} + O\left(\epsilon^2\right)\right) s, s \right\rangle_{\widehat{N}_\epsilon \setminus \cup_{p' \in \pi^{-1}(p)} B_{p'}^{\widehat{N}_\epsilon}(a_0)}, \tag{1.16}$$

which implies (when $\epsilon > 0$ is small enough) $\ker(D_\alpha^\xi) = \{0\}$ (cf. [3, Theorem 8.2]), which contradicts (1.4) where the right hand side is nonzero. This completes the proof of Theorem 0.2 for the case where N is a compactly enlargeable manifold.

For the general case where $\widehat{N}_\epsilon$ is noncompact, one can combine the above arguments with the method in [8] to complete the proof of Theorem 0.2. We leave the details to the readers who have interest in this.

Acknowledgements. This work was partially supported by NNSFC.

References

[1] M. F. Atiyah and I. M. Singer, *The index of elliptic operators I*, Ann. of Math. 87 (1968), 484-530.

[2] J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 74, ii+298 pp.

[3] B. Booß-Bavnbek and K. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1993.

[4] M. Gromov and H. B. Lawson, *Spin and scalar curvature in the presence of a fundamental group I*, Ann. of Math. 111 (1980), 209-230.

[5] M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. I.H.E.S. 58 (1983), 295-408.

[6] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, NJ, 1989.

[7] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris, Série A, 257 (1963), 7-9.

[8] W. Zhang, *Positive scalar curvature on foliations: the enlargeability*, Preprint, arXiv:1703.04313.