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AN L^2-ALEXANDER INVARIANT FOR KNOTS

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In this paper, in an attempt to extend an earlier work of Lück, we construct a knot invariant with parameter in $\mathbf{C}^*$ by using the fundamental L^2-representation of the fundamental group of the knot complement, which may be thought of as an L^2-analogue of the usual Alexander polynomial of the knot in S^3. When restricted to $U(1)$ parameters, we interpret this invariant in terms of the $U(1)$ twisted L^2-Reidemeister torsion. We also show that this L^2-invariant depends only on the norm $|t|$ for $t \in \mathbf{C}^*$. In particular, this implies an unexpected rigidity property of the $U(1)$ twisted L^2-torsion on a knot complement. A possible relationship with the volume conjecture is discussed.

Keywords: Knot; Fuglede–Kadison determinant; Alexander invariant; L^2-Reidemeister torsion; volume conjecture.

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1. Introduction

It is well known that the Alexander polynomial plays an important role in the theory of knots, and that Milnor [22] first interpreted it in terms of the Reidemeister torsion of the knot complement. Twisted versions of the Alexander polynomial have been introduced and studied by Lin [13] *et al.* (cf. [9, 11, 12, 27]). Moreover, Kitano [12] interprets these twisted invariants in terms of Reidemeister torsions along the lines in [22]. The common feature of these works is that one constructs knot invariants with parameters through finite-dimensional representations of the knot group.

In this paper, we construct knot invariants with parameters through infinite-dimensional representations of the knot group. Certainly, there are many such representations. We concentrate on the most natural one which is associated with

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the universal covering of the knot complement and the $\mathbf{C}^*$ representations of the first homology group of the knot complement. We call such an invariant the L^2-Alexander invariant. It extends the one constructed by Lück [17]. In [17], the $\mathbf{C}^*$ representation is the trivial one of the first homology group of the knot complement.

We expect that such L^2-invariants with parameters and their twisted versions would play a role in the study of knots. For $U(1)$ representations, our L^2-invariant is related to the (twisted) L^2-Reidemeister torsion of the knot complement (see also [22, 12, 17, 20]). This extends the corresponding result of Lück for the trivial representation case, with a different proof. Moreover, we prove a surprising result that this L^2-invariant is rigid with respect to $U(1)$ parameters. In the case of hyperbolic knot, this indicates that the hyperbolic volume of the knot complement has certain $U(1)$ rigidity. It might be helpful in the way of understanding the volume conjecture (cf. [10] and [24]) in knot theory.

This paper is organized as follows. In Sec. 2, recall the basic properties of the Fuglede–Kadison determinant for morphisms between free Hilbert modules over a group von Neumann algebra. In Sec. 3, we construct the L^2-Alexander invariant with $U(1)$-parameters through the Wirtinger presentations of a knot. In Secs. 4 and 5, we interpret the L^2-Alexander invariant through the L^2-Reidemeister torsion of the knot complement. We prove the $U(1)$ rigidity of the L^2-Alexander invariant in Sec. 6, and we show how to define the L^2-Alexander invariant with $\mathbf{C}^*$ parameters in Sec. 7. Possible relations with the volume conjecture are discussed in Sec. 8.

We will try to present the L^2-theory involved in an elementary form which we hope to be helpful to those who are not familiar with this technique.

2. Group von Neumann Algebra and Fuglede–Kadison Determinant

In this section, we review the definition and basic properties of the Fuglede–Kadison determinant introduced in [5]. The basic reference is the comprehensive book of Lück [16].

Let Γ be a finitely generated discrete group. We assume that Γ is an infinite group. Let $l^2(\Gamma)$ be the standard Hilbert space of squared summable formal sums over Γ with complex coefficients. An element in $l^2(\Gamma)$ can be written as

$$a = \sum_{\gamma\in\Gamma} a_\gamma \gamma, \quad a_\gamma \in \mathbf{C}, \quad \text{with} \quad \sum_{\gamma\in\Gamma} |a_\gamma|^2 < +\infty.$$

If $a = \sum_{\gamma\in\Gamma} a_\gamma\gamma$ and $b = \sum_{\gamma\in\Gamma} b_\gamma\gamma$ are two elements in $l^2(\Gamma)$, their inner product is given by $\langle a, b\rangle = \sum_{\gamma\in\Gamma} a_\gamma \overline{b}_\gamma$.

The left multiplication with elements in Γ defines a natural unitary action of Γ on $l^2(\Gamma)$. The group von Neumann algebra $\mathcal{N}(\Gamma)$ is the algebra of Γ-equivariant bounded linear operators from $l^2(\Gamma)$ to $l^2(\Gamma)$. The von Neumann trace on $\mathcal{N}(\Gamma)$ is defined by

$$\mathrm{Tr}_\tau : \mathcal{N}(\Gamma) \to \mathbf{C}, \quad f \mapsto \langle f(e), e\rangle, \tag{2.1}$$

where $e \in \Gamma \subset l^2(\Gamma)$ is the unit element. The right multiplication induces a natural action of Γ on $l^2(\Gamma)$ commuting with the left multiplication of Γ. Thus, $\Gamma \subset \mathcal{N}(\Gamma)$. Moreover, for any $\gamma \in \Gamma \subset \mathcal{N}(\Gamma)$,

$$\mathrm{Tr}_\tau[\gamma] = 1 \quad \text{if } \gamma = e; \quad \mathrm{Tr}_\tau[\gamma] = 0 \quad \text{if } \gamma \neq e. \tag{2.2}$$

For any positive integer n, set

$$l^2(\Gamma)^{[n]} = \underbrace{l^2(\Gamma) \oplus \cdots \oplus l^2(\Gamma)}_{n}.$$

We call it a free $\mathcal{N}(\Gamma)$-Hilbert module of rank n. The action of Γ on $l^2(\Gamma)$ (through left multiplications) induces a canonical action of Γ on $l^2(\Gamma)^{[n]}$. A morphism between two free $\mathcal{N}(\Gamma)$-Hilbert modules is a Γ-equivariant bounded linear map between them.

Let $f : l^2(\Gamma)^{[n]} \to l^2(\Gamma)^{[n]}$ be such a morphism. Let e_i $(i = 1, \ldots, n)$ be the unit element in the ith copy of $l^2(\Gamma)$ in $l^2(\Gamma)^{[n]}$. Then, we can extend the von Neumann trace in (2.1) to define

$$\mathrm{Tr}_\tau[f] = \sum_{i=1}^{n} \langle f(e_i), e_i \rangle. \tag{2.3}$$

The Fuglede–Kadison determinant $\mathrm{Det}_\tau(f)$ of f can be defined as follows:

(i) If f is invertible and f^* is the adjoint of f, then define (cf. [5, Definition] and [16, Lemma 3.15(2)])

$$\mathrm{Det}_\tau(f) = \exp\left(\frac{1}{2}\mathrm{Tr}_\tau[\log(f^* f)]\right); \tag{2.4}$$

(ii) If f is injective, then define (cf. [5, Lemma 5] and [16, Lemma 3.15(4) and (5)])

$$\mathrm{Det}_\tau(f) = \lim_{\varepsilon \to 0^+} \sqrt{\mathrm{Det}_\tau(f^* f + \varepsilon)} = \sqrt{\mathrm{Det}_\tau(f^* f)}. \tag{2.5}$$

If f is injective and $\mathrm{Det}_\tau(f) \neq 0$, then we say that f is of determinant class. From (2.2) and (2.4), one sees easily that for any $\lambda \in \mathbf{C}$ with $\lambda \neq 0$,

$$\mathrm{Det}_\tau(\lambda \mathrm{Id} : l^2(\Gamma) \to l^2(\Gamma)) = |\lambda|. \tag{2.6}$$

We also recall the following basic properties of Det_τ.

(iii) If both $f, g : l^2(\Gamma)^{[n]} \to l^2(\Gamma)^{[n]}$ are injective, then $g \circ f$ is also injective. Moreover (cf. [5, Lemma 5 and the discussion on p. 529] and [16, Theorem 3.14(1)]),

$$\mathrm{Det}_\tau(g \circ f) = \mathrm{Det}_\tau(f) \cdot \mathrm{Det}_\tau(g). \tag{2.7}$$

(iv) If $f_1 : l^2(\Gamma)^{[n]} \to l^2(\Gamma)^{[n]}$, $f_2 : l^2(\Gamma)^{[m]} \to l^2(\Gamma)^{[m]}$ and $f_3 : l^2(\Gamma)^{[m]} \to l^2(\Gamma)^{[n]}$ are morphisms such that f_1 and f_2 are injective. Then, $\left(\begin{smallmatrix} f_1 & 0 \\ f_3 & f_2 \end{smallmatrix}\right) : l^2(\Gamma)^{[n+m]} \to l^2(\Gamma)^{[n+m]}$ is also injective and (cf. [2, Theorem 1.10(f)] and [16, Theorem 3.14(2)]),

$$\mathrm{Det}_\tau \begin{pmatrix} f_1 & 0 \\ f_3 & f_2 \end{pmatrix} = \mathrm{Det}_\tau(f_1) \cdot \mathrm{Det}_\tau(f_2). \tag{2.8}$$

There is an alternative definition of the Fuglede–Kadison determinant for invertible morphisms. Let $f : l^2(\Gamma)^{[n]} \to l^2(\Gamma)^{[n]}$ be an invertible morphism. Then, there exists a C^1 path f_u, $u \in [0,1]$, of invertible morphisms such that $f_0 = f$, $f_1 = \mathrm{Id}$, and (cf. [5, Theorem 1 and Lemma 2] and [2, Theorem 1.10(e)])

$$\log(\mathrm{Det}_\tau(f)) = -\mathrm{Re}\biggl(\int_0^1 \mathrm{Tr}_\tau \left[f_u^{-1} \frac{df_u}{du} \right] du \biggr). \tag{2.9}$$

3. The L^2-Alexander Invariant of a Knot

Let $K \subset S^3$ be a knot. Let $\Gamma = \pi_1(S^3 \setminus K)$ denote the associated knot group. A Wirtinger presentation of Γ is given by

$$P(\Gamma) = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle. \tag{3.1}$$

Let $F_k = \langle x_1, \ldots, x_k \rangle$ be the free group of rank k, and $\phi : F_k \to \Gamma$ denote the canonical surjective homomorphism associated to (3.1). Then, it induces a ring homomorphism

$$\widetilde{\phi} : \mathbf{Z}[F_k] \to \mathbf{Z}[\Gamma]. \tag{3.2}$$

Define α to be the canonical abelianization

$$\alpha : \Gamma \to U(1), \quad \alpha(x_1) = \cdots = \alpha(x_k) = t \in U(1) \subset \mathbf{C}. \tag{3.3}$$

Let $GL(l^2(\Gamma))$ denote the set of invertible elements in $\mathcal{N}(\Gamma)$. Let

$$\rho_\Gamma : \Gamma \to GL(l^2(\Gamma)) \tag{3.4}$$

denote the fundamental representation of Γ, which is given by the right multiplication of the elements in Γ. The tensor product representation $\rho \otimes \alpha$ induces a ring homomorphism of the integral group rings

$$\widetilde{\rho_\Gamma \otimes \alpha} : \mathbf{Z}[\Gamma] \to \mathcal{N}(\Gamma) \otimes \mathbf{Z}[t^{\pm 1}] \subset \mathcal{N}(\Gamma). \tag{3.5}$$

Let the composition of the ring homomorphism in (3.2) with the homomorphism in (3.5) be denoted by

$$\Psi = (\widetilde{\rho_\Gamma \otimes \alpha}) \circ \widetilde{\phi} : \mathbf{Z}[F_k] \to \mathcal{N}(\Gamma). \tag{3.6}$$

Consider the morphism

$$A_{\rho_\Gamma \otimes \alpha} : l^2(\Gamma)^{[k-1]} \to l^2(\Gamma)^{[k]} \tag{3.7}$$

which when written in the $(k-1)\times k$ matrix form, the (i,j)-component is given by

$$A_{\rho_\Gamma\otimes\alpha,(i,j)} = \Psi\left(\frac{\partial r_i}{\partial x_j}\right) \in \mathcal{N}(\Gamma)\otimes \mathbf{Z}[t^{\pm 1}] \subset \mathcal{N}(\Gamma), \tag{3.8}$$

where $\frac{\partial r_i}{\partial x_j}$ is the standard Fox derivative.

We call $A_{\rho_\Gamma\otimes\alpha}$ the L^2-Alexander matrix of the presentation $P(\Gamma)$ associated to the fundamental representation ρ_Γ and the representation α.

For any $1 \le j \le k$, let

$$A^j_{\rho_\Gamma\otimes\alpha} : l^2(\Gamma)^{[k-1]} \to l^2(\Gamma)^{[k-1]} \tag{3.9}$$

denote the morphism obtained from $A_{\rho_\Gamma\otimes\alpha}$ by removing the jth column from its matrix form. The following result can be thought of as an L^2-analogue of [27, Lemmas 2 and 3].

Lemma 3.1. (i) *For any* $1 \le j \le k$, $\Psi(x_j - 1) \in \mathcal{N}(\Gamma)$ *is injective and has dense image.*

(ii) *If one of the* $A^j_{\rho_\Gamma\otimes\alpha}$*'s,* $1 \le j \le k$, *is injective, then every* $A^j_{\rho_\Gamma\otimes\alpha}$, $1 \le j \le k$, *is injective. Moreover, for any* $1 \le j < j' \le k$, *one has*

$$\mathrm{Det}_\tau\left(A^j_{\rho_\Gamma\otimes\alpha}\right)\mathrm{Det}_\tau(\Psi(x_{j'}-1)) = \mathrm{Det}_\tau\left(A^{j'}_{\rho_\Gamma\otimes\alpha}\right)\mathrm{Det}_\tau(\Psi(x_j-1)). \tag{3.10}$$

Proof. (i) From (3.2), (3.3) and (3.6), one sees that

$$\Psi(x_j - 1) = t\rho_\Gamma(\phi(x_j)) - \mathrm{Id}. \tag{3.11}$$

Assume $(t\rho_\Gamma(\phi(x_j))-\mathrm{Id})a = 0$ for $a = \sum_{\gamma\in\Gamma} a_\gamma\gamma \in l^2(\Gamma)$. It is a straight-forward calculation that for any $\gamma' \in \Gamma$ and $p \in \mathbf{Z}$,

$$t a_{\gamma'\gamma_j^{p-1}} = a_{\gamma'\gamma_j^p}. \tag{3.12}$$

For $t \in U(1)$, we have

$$|a_{\gamma'}| = |a_{\gamma'\gamma_j^p}|, \quad p \in \mathbf{Z}. \tag{3.13}$$

Note that $\gamma_j = \phi(x_j) \in \Gamma$ is of infinite order. By (3.13) and the L^2-condition $\sum_{\gamma\in\Gamma}|a_\gamma|^2 < +\infty$, one gets $a_{\gamma'} = 0$. Thus, we obtain

$$\ker(t\rho_\Gamma(\phi(x_j)) - \mathrm{Id}) = \{0\}. \tag{3.14}$$

Hence, $\Psi(x_j - 1) \in \mathcal{N}(\Gamma)$ is injective and thus, also has dense image (see [4]). Indeed, the dense image property follows from (3.14) immediately as one has

$$(\mathrm{Im}(t\rho_\Gamma(\phi(x_j)) - 1))^\perp = \ker\left(\bar{t}\rho_\Gamma\left(\phi\left(x_j^{-1}\right)\right) - 1\right) = \{0\}. \tag{3.15}$$

(ii) Without loss of generality, we may assume $j = 1$ and $j' = 2$. It is easy to see that

$$\sum_{l=1}^{k} \frac{\partial r_i}{\partial x_l}(x_l - 1) = r_i - 1 \quad \text{in } \mathbf{Z}[F_k], \tag{3.16}$$

from which we get

$$\sum_{l=1}^{k} \Psi\left(\frac{\partial r_i}{\partial x_l}\right) \Psi(x_l - 1) = 0. \tag{3.17}$$

Let $A_1,\ A_2 : l^2(\Gamma)^{[k-1]} \to l^2(\Gamma)^{[k-1]}$ be the endomorphisms such that

$$A_1 = \Psi(x_1 - 1)|_{l^2(\Gamma)} \oplus \mathrm{Id}|_{l^2(\Gamma)} \oplus \cdots \oplus \mathrm{Id}|_{l^2(\Gamma)}, \tag{3.18}$$

and A_2, as a $(k-1) \times (k-1)$ matrix form, is determined by

$$A_{2,(l,1)} = \Psi(x_{l+1} - 1), \quad A_{2,(i,j)} = \delta_{ij} \cdot (\mathrm{Id} : l^2(\Gamma) \to l^2(\Gamma)) \quad \text{for } j \geq 2. \tag{3.19}$$

From (3.17)–(3.19) and the definition of $A^j_{\rho_\Gamma \otimes \alpha}$, one deduces that

$$A^1_{\rho_\Gamma \otimes \alpha} \cdot A_2 = -A^2_{\rho_\Gamma \otimes \alpha} \cdot A_1. \tag{3.20}$$

By (i), it is clear that both A_1 and A_2 are injective. Thus, $A^1_{\rho_\Gamma \otimes \alpha}$ is injective if and only if $A^2_{\rho_\Gamma \otimes \alpha}$ is injective. By (2.8), one finds

$$\mathrm{Det}_\tau(A_j) = \mathrm{Det}_\tau(\Psi(x_j - 1)), \quad j = 1, 2. \tag{3.21}$$

From (2.7), (3.20) and (3.21), the result follows. □

Proposition 3.2. *For any $1 \leq j \leq k$, the following identity holds,*

$$\mathrm{Det}_\tau(\Psi(x_j - 1)) = 1. \tag{3.22}$$

Proof. By (3.11), $\Psi(x_j - 1) = t\rho_\Gamma(\phi(x_j)) - \mathrm{Id} = t\rho_\Gamma(\gamma_j) - \mathrm{Id}$. First of all, we observe that for any $t \in \mathbf{C}$ with $|t| \neq 1$, $t\rho_\Gamma(\gamma_j) - \mathrm{Id}$ is invertible in $\mathcal{N}(\Gamma)$. Indeed, if $|t| < 1$, then $\|t\rho_\Gamma(\gamma_j)\| < 1$, one clearly sees that $\mathrm{Id} - t\rho_\Gamma(\gamma_j)$ is invertible; if $|t| > 1$, $\mathrm{Id} - t\rho_\Gamma(\gamma_j) = -t\rho_\Gamma(\gamma_j)(\mathrm{Id} - t^{-1}\rho_\Gamma(\gamma_j)^{-1})$, then $\mathrm{Id} - t\rho_\Gamma(\gamma_j)$ is also invertible.

For any $t \in \mathbf{C}$ with $|t| < 1$ and $u \in [0,1]$, $B_u(t) = \mathrm{Id} - ut\rho_\Gamma(\gamma_j)$ is also invertible. Set

$$f(t) = \int_0^1 \mathrm{Tr}_\tau\left[B_u(t)^{-1}\frac{dB_u(t)}{du}\right]du = -\int_0^1 \mathrm{Tr}_\tau\left[(\mathrm{Id} - ut\rho_\Gamma(\gamma_j))^{-1}t\rho_\Gamma(\gamma_j)\right]du. \tag{3.23}$$

By (3.23) and the fact that $\rho_\Gamma(\gamma_j)$ is of infinite order, we have

$$\begin{aligned} f(t) &= -\int_0^1 \sum_{i=0}^{+\infty} \mathrm{Tr}_\tau\left[(ut\rho_\Gamma(\gamma_j))^i t\rho_\Gamma(\gamma_j)\right]du \\ &= -\sum_{i=0}^{+\infty} \frac{t^{i+1}}{i+1}\mathrm{Tr}_\tau\left[\rho_\Gamma(\gamma_j)^{i+1}\right] = 0, \end{aligned} \tag{3.24}$$

for $t \in \mathbf{C}$ with $|t| < 1$. By (2.9) and (3.24), we get

$$\log(\mathrm{Det}_\tau(\mathrm{Id} - t\rho_\Gamma(\gamma_j))) = \mathrm{Re}(f(t)) = 0. \tag{3.25}$$

In order to prove (3.22), we must show that (3.25) still holds when $|t| = 1$. Simply denote $\rho_\Gamma(\gamma_j)$ by γ. For any $\varepsilon > 0$ and $t \in U(1)$, we have

$$\begin{aligned}((1+\varepsilon)\mathrm{Id} - t\gamma)^*((1+\varepsilon)\mathrm{Id} - t\gamma) &= \big((1+\varepsilon)\mathrm{Id} - t^{-1}\gamma^{-1}\big)((1+\varepsilon)\mathrm{Id} - t\gamma)\\ &= (1+\varepsilon)\left(2\,\mathrm{Id} - \big(t\gamma + t^{-1}\gamma^{-1}\big) + \frac{\varepsilon^2}{1+\varepsilon}\mathrm{Id}\right).\end{aligned} \tag{3.26}$$

By Lemma 3.1, we know that $2\,\mathrm{Id} - (t\gamma + t^{-1}\gamma^{-1}) = (\mathrm{Id} - t\gamma)^*(\mathrm{Id} - t\gamma)$ is an injective positive operator. By applying (2.5), we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \mathrm{Det}_\tau\left(2\,\mathrm{Id} - \big(t\gamma + t^{-1}\gamma^{-1}\big) + \frac{\varepsilon^2}{1+\varepsilon}\mathrm{Id}\right)\\ &\quad= \mathrm{Det}_\tau\big(2\,\mathrm{Id} - \big(t\gamma + t^{-1}\gamma^{-1}\big)\big)\\ &\quad= \mathrm{Det}_\tau((\mathrm{Id} - t\gamma)^*(\mathrm{Id} - t\gamma))\\ &\quad= (\mathrm{Det}_\tau(\mathrm{Id} - t\gamma))^2.\end{aligned} \tag{3.27}$$

On the other hand, since $|t| = 1$, by (2.6) and (3.25),

$$\begin{aligned}&\mathrm{Det}_\tau(((1+\varepsilon)\mathrm{Id} - t\gamma)^*((1+\varepsilon)\mathrm{Id} - t\gamma))\\ &\quad= (1+\varepsilon)^2\left(\mathrm{Det}_\tau\left(\mathrm{Id} - \frac{t}{1+\varepsilon}\gamma\right)\right)^2 = (1+\varepsilon)^2.\end{aligned} \tag{3.28}$$

By (3.26)–(3.28), one gets $\mathrm{Det}_\tau(\mathrm{Id} - t\gamma) = 1$ for $t \in U(1)$. Hence, the result follows. $\square$

Remark 3.3. One can indeed prove that for any $t \in \mathbf{C}$ and an element $\gamma \in \Gamma$ which is of infinite order, then $\mathrm{Id} - t\gamma : l^2(\Gamma) \to l^2(\Gamma)$ is injective and has dense image. Moreover,

$$\mathrm{Det}_\tau(\mathrm{Id} - t\gamma) = \max(1, |t|).$$

This shows that the Fuglede–Kadison determinant is closely related to the Mahler measure (compare with [16, (3.23)] and [23]) of a polynomial.

Recall the following types of transformations for group presentations: (Ia) To replace one of the relators r_i by its inverse r_i^{-1}; (Ib) To replace one of the relators r_i by its conjugate wr_iw^{-1} ($w \in F_k$); (Ic) To replace one of the relators r_i by r_ir_k ($i \neq k$); (II) To add a new generator x and a new relator xw^{-1} for any word w in terms of previous generators. Two presentations are strongly Tietze-equivalent if there is a finite sequence of operations of transformation types (Ia), (Ib), (Ic) and (II) and their inverses such that one presentation can be transformed to another one.

The following result may be thought of as an L^2-analogue of [27, Theorem 2].

Proposition 3.4. *The quantity*

$$\Delta_K^{(2)}(t) = \mathrm{Det}_\tau\big(A^1_{\rho_\Gamma\otimes\alpha}\big) \tag{3.29}$$

does not depend on the choice of the Wirtinger presentation $P(\Gamma)$ in (3.1).

Proof. By [27, Lemma 6], all Wirtinger presentations of Γ are strongly Tietze-equivalent. We need to show that $\mathrm{Det}_\tau\big(A^1_{\rho_\Gamma\otimes\alpha}\big)$ is invariant under the transformations defining the strong Tietze equivalence. This can be carried out in the same way as in [27, Proof of Theorem 2]. We present the proof for completeness.

Type (Ia). This amounts to change the homomorphism $A_{\rho_\Gamma\otimes\alpha}$ to $-A_{\rho_\Gamma\otimes\alpha}$, which clearly does not change $\mathrm{Det}_\tau\big(A^1_{\rho_\Gamma\otimes\alpha}\big)$.

Type (Ib). Note that $\Psi\big(\frac{\partial}{\partial x_j}\big(wr_iw^{-1}\big)\big) = \Psi(w)\Psi\big(\frac{\partial r_i}{\partial x_j}\big)(1 \leq j \leq k)$. By proceeding as in the proof of Lemma 2.1(ii), we have that $\mathrm{Det}_\tau(A^1_{\rho_\Gamma\otimes\alpha})$ changes by a factor $\mathrm{Det}_\tau(\Psi(w))$. By (3.3) and (3.6), $\Psi(w) \in GL(l^2(\Gamma))$ is a unitary operator with $\mathrm{Det}_\tau(\Psi(w)) = 1$. Thus the type (Ib) transformation does not change $\mathrm{Det}_\tau(A^1_{\rho_\Gamma\otimes\alpha})$.

Type (Ic). For this transformation we have

$$\Psi\left(\frac{\partial}{\partial x_j}(r_ir_m)\right) = \Psi\left(\frac{\partial r_i}{\partial x_j}\right) + \Psi\left(\frac{\partial r_m}{\partial x_j}\right), \quad 1 \leq j \leq k.$$

Let $B : l^2(\Gamma)^{[k-1]} \to l^2(\Gamma)^{[k-1]}$ be the endomorphism, as a $(k-1)\times(k-1)$ matrix, which takes the form $B_{i,m} = \mathrm{Id}$, and $B_{s,t} = \delta_{st}\cdot(\mathrm{Id} : l^2(\Gamma) \to l^2(\Gamma))$ otherwise. Then, $A^1_{\rho_\Gamma\otimes\alpha}$ changes to $BA^1_{\rho_\Gamma\otimes\alpha}$ under the transformation (Ic). Note that $\mathrm{Det}_\tau(B) = 1$. Thus, the type (Ic) transformation does not change $\mathrm{Det}_\tau(A^1_{\rho_\Gamma\otimes\alpha})$.

Type (II). In this case, the corresponding endomorphism $A'^1_{\rho_\Gamma\otimes\alpha} : l^2(\Gamma)^{[k-1]} \to l^2(\Gamma)^{[k-1]}$ can be written as $A^1_{\rho_\Gamma\otimes\alpha} \oplus \mathrm{Id}_{l^2(\Gamma)}$ plus a mapping from $l^2(\Gamma)^{[k-1]}$ to $l^2(\Gamma)$. Thus, by (2.8),

$$\mathrm{Det}_\tau\big(A'^1_{\rho_\Gamma\otimes\alpha}\big) = \mathrm{Det}_\tau\big(A^1_{\rho_\Gamma\otimes\alpha}\big). \tag{3.30}$$

The type (II) transformation does not change $\mathrm{Det}_\tau\big(A^1_{\rho_\Gamma\otimes\alpha}\big)$. Similarly, the inverse of the transformation (Ia), (Ib), (Ic) or (II) does not change $\mathrm{Det}_\tau\big(A^1_{\rho_\Gamma\otimes\alpha}\big)$.

Therefore, $\Delta^{(2)}_K(t)$ is invariant under the strong Tietze-equivalent transformations. Hence, the result follows. □

By Lemma 3.1, Propositions 3.2 and 3.4, we see that $\Delta^{(2)}_K(t)$ depends only on the knot K and the parameter $t \in U(1)$. Since its construction is closely related to the usual construction of the Alexander polynomial as well as the twisted Alexander polynomial of a knot, we make the following definition.

Definition 3.5. We define $\Delta^{(2)}_K(t)$ to be the L^2-Alexander invariant of the knot K in S^3.

Remark 3.6. It is clear that $\Delta^{(2)}_K(t)$ can be defined by using any presentation of Γ which is strongly Tietze-equivalent to some Wirtinger presentation of Γ.

Remark 3.7. When $t = 1$, $\Delta^{(2)}_K(t)$ has been studied by Lück [17] (cf. [16, Theorem 4.9]), who shows that $\Delta^{(2)}_K(1)$ is equivalent to the L^2-Reidemeister torsion of $S^3\backslash K$.

In the next two sections, we interpret $\Delta_K^{(2)}(t)$ with $t \in U(1)$ as certain twisted L^2-Reidemeister torsion of $S^3\backslash K$.

Remark 3.8. In view of [27, Sec. 5], the above construction can also be applied to links.

The following result can be thought of as a complement to Remark 3.3, and it is true for any finitely generated infinite group Γ (not necessarily the knot group).

Proposition 3.9.[a] *If* $\gamma \in \Gamma$ *is of finite order* $N > 0$, *and* $t \in \mathbf{C}$ *satisfies* $t^N \neq 1$, *then* $\mathrm{Id} - t\gamma : l^2(\Gamma) \to l^2(\Gamma)$ *is injective and*

$$\mathrm{Det}_\tau(\mathrm{Id} - t\gamma) = |1 - t^N|^{\frac{1}{N}}. \tag{3.31}$$

Proof. If $|t| < 1$, then $\mathrm{Id} - t\gamma$ is invertible and one can define $f(t)$ as in (3.23), with $\rho_\Gamma(\gamma_j)$ there replaced by γ. Since γ is of order N, by (3.24),

$$f(t) = -\sum_0^{+\infty} \frac{t^{i+1}}{i+1}\mathrm{Tr}_\tau[\gamma^{i+1}] = -\sum_{j=1}^{+\infty} \frac{t^{Nj}}{Nj} = \frac{\log(1-t^N)}{N}. \tag{3.32}$$

By (2.9) and (3.32), $\log(\mathrm{Det}_\tau(\mathrm{Id} - t\gamma)) = \mathrm{Re}(f(t)) = \frac{\log|1-t^N|}{N}$, and (3.31) follows.

If $|t| > 1$, then by (3.32), we have

$$\mathrm{Det}_\tau(\mathrm{Id} - t\gamma) = |t|\left|1 - \left(\frac{1}{t}\right)^N\right|^{\frac{1}{N}} = |1 - t^N|^{\frac{1}{N}}. \tag{3.33}$$

Now, assume $|t| = 1$ with $t^N \neq 1$. If $a = \sum_{\gamma' \in \Gamma} a_{\gamma'}\gamma' \in l^2(\Gamma)$ satisfies $(\mathrm{Id} - t\gamma)a = 0$, then by (3.12), we have $t^N a_{\gamma'} = a_{\gamma'\gamma^N} = a_{\gamma'}$ for any $\gamma' \in \Gamma$, which implies $a_{\gamma'} = 0$. Thus, $a = 0$ which implies that $\mathrm{Id} - t\gamma$ is injective. By proceeding as in (3.26)–(3.28), the result follows. □

4. The L^2-Reidemeister Torsion

We recall the definition of the L^2-Reidemeister torsion ([3, 20, 15]).

Let (C_*, ∂) be a finite length $\mathcal{N}(\Gamma)$-chain complex

$$(C_*, \partial) : 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0, \tag{4.1}$$

where each C_i $(0 \leq i \leq n)$ is a (finite rank) $\mathcal{N}(\Gamma)$ free Hilbert module. Assume that (C_*, ∂) is weakly acyclic, i.e. for any $0 \leq i \leq n$, $\ker(\partial_i) = \overline{\mathrm{Im}(\partial_{i-1})}$ (usually one uses the terminology "L^2-acyclic").

Let $\partial_i^* : C_{i-1} \to C_i$ be the adjoint of $\partial_i : C_i \to C_{i-1}$. Then, $\partial_i\partial_i^* : \overline{\mathrm{Im}(\partial_i)} \to \overline{\mathrm{Im}(\partial_i)}$ is injective $(0 \leq i \leq n)$.

[a]This result has also been obtained independently by Xiao-Song Lin.

We call (C_*, ∂) is of determinant class if for any $0 \le i \le n$, $\partial_i \partial_i^* : \overline{\mathrm{Im}(\partial_i)} \to \overline{\mathrm{Im}(\partial_i)}$ is of determinant class. In this case, the L^2-Reidemeister torsion of (C_*, ∂) is defined to be a real number $T^{(2)}(C_*, \partial)$ given by (cf. [16, Definition 3.29])

$$\log T^{(2)}(C_*, \partial) = -\frac{1}{2} \sum_{i=0}^{n} (-1)^i \log \mathrm{Det}_\tau \big(\partial_i \partial_i^* |_{\overline{\mathrm{Im}(\partial_i)}}\big). \tag{4.2}$$

Let $\rho : \pi_1(X) \to GL(H)$ be an $\mathcal{N}(\Gamma)$-linear representation of $\Gamma = \pi_1(X)$ on a (finite rank) free $\mathcal{N}(\Gamma)$ Hilbert module, where X is a finite cell complex. Let $\widetilde{X}$ be the universal covering of X. Thus, the chain complex $(C_*(\widetilde{X}) \otimes H, \widetilde{\partial})$ induces canonically a chain complex $(C_*(X, H_\rho), \partial_\rho)$ in the sense of (4.1) with $C_*(X, H_\rho) = (C_*(\widetilde{X}) \otimes_{\pi_1(X),\rho} H)$.

If $(C_*(X, H_\rho), \partial_\rho)$ is weakly acyclic and of determinant class, then we have its L^2-Reidemeister torsion $T^{(2)}(C_*(X, H_\rho), \partial_\rho)$ as in (4.2).

Remark 4.1. If $\rho : \pi_1(X) \to GL(H)$ is unitary, then $T^{(2)}(C_*(X, H_\rho), \partial_\rho)$ is a well defined piecewise linear invariant.

5. The L^2-Alexander Invariant as an L^2-Reidemeister Torsion

Let W be a 2-dimensional cell complex constructed from one 0-cell p, k 1-cells $x_1, \ldots, x_k$ and $(k-1)$ 2-cells $D_1, \ldots, D_{k-1}$ with attaching maps given by $r_1, \ldots, r_{k-1}$. It is well known that the knot complement $S^3 \backslash K$ collapses to the 2-dimensional complex W. Let

$$\rho_\alpha = \rho \otimes \alpha : \Gamma \to GL(l^2(\Gamma)) \otimes U(1) = GL(l^2(\Gamma)) \tag{5.1}$$

denote the representation of Γ obtained from the tensor product of the representations in (3.3) and (3.4).

Then, the $\mathcal{N}(\Gamma)$-chain complex $(C_*(S^3 \backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is weakly acyclic if and only if $(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is weakly acyclic. Moreover, by the simple homotopy invariance of the L^2-Reidemeister torsion (cf. [18, Corollary 3.12]), $(C_*(S^3 \backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is of determinant class if and only if $(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is of determinant class, and in this case,

$$T^{(2)}(C_*(S^3 \backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha}) = T^{(2)}\big(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha}\big). \tag{5.2}$$

Proposition 5.1. *The complex $(C_*(S^3 \backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is weakly acyclic if and only if $A^1_{\rho_\Gamma \otimes \alpha}$ in* (3.9) *is injective. Moreover, $A^1_{\rho_\Gamma \otimes \alpha}$ is of determinant class if and only if $(C_*(S^3 \backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is of determinant class and*

$$T^{(2)}\big(C_*(S^3 \backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha}\big) = \frac{1}{\Delta_K^{(2)}(t)}. \tag{5.3}$$

Proof. By (5.2), we need only to compute the L^2-Reidemeister torsion of $(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$. For any $m \in \mathbf{N}$, let $l^2(\Gamma)^{[m]}_{\rho_\alpha}$ denote the $\mathcal{N}(\Gamma)$-Hilbert module of rank m,

$$l^2(\Gamma)^{[m]}_{\rho_\alpha} = \underbrace{l^2(\Gamma)_{\rho_\alpha} \oplus \cdots \oplus l^2(\Gamma)_{\rho_\alpha}}_{m}. \tag{5.4}$$

Then, the chain complex $(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is given by,

$$0 \to l^2(\Gamma)^{[k-1]}_{\rho_\alpha} \overset{\partial_2}{\to} l^2(\Gamma)^{[k]}_{\rho_\alpha} \overset{\partial_1}{\to} l^2(\Gamma)_{\rho_\alpha} \to 0, \tag{5.5}$$

where ∂_2, as a $(k-1) \times k$ matrix (with respect to the right matrix multiplications), is given by

$$\partial_2 = A_{\rho\otimes\alpha} = \left(\Psi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{(k-1)\times k} \quad \text{with} \quad 1 \le i \le k-1, \quad 1 \le j \le k, \tag{5.6}$$

and ∂_1 is given by

$$\partial_1 = (\Psi(x_1 - 1), \ldots, \Psi(x_k - 1))^t. \tag{5.7}$$

By Lemma 3.1 and (5.7), the L^2-homology is given by

$$H_0^{(2)}(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha}) = l^2(\Gamma)_{\rho_\alpha} / \overline{\mathrm{Im}(\partial_1)} = 0.$$

On the other hand, it is clear that

$$\begin{aligned} &\mathrm{rk}^{(2)}(H_0^{(2)}(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})) - \mathrm{rk}^{(2)}(H_1^{(2)}(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})) \\ &\quad + \mathrm{rk}^{(2)}(H_2^{(2)}(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})) = 1 - k + (k-1) = 0, \end{aligned} \tag{5.8}$$

where $\mathrm{rk}^{(2)}$ is the notation of von Neumann rank (dimension).

From (5.7) and (5.8), one sees that $(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is weakly acyclic if and only if ∂_2 is injective (i.e. if and only if $\mathrm{rk}^{(2)}(H_2^{(2)}(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})) = 0$).

Let $A' : l^2(\Gamma)^{[k]}_{\rho_\alpha} \to l^2(\Gamma)^{[k]}_{\rho_\alpha}$ be the $k \times k$ matrix such that

$$A'_{i,1} = \Psi(x_i - 1), \quad A'_{i,j} = \delta_{ij}\{\mathrm{Id} : l^2(\Gamma)_{\rho_\alpha} \to l^2(\Gamma)_{\rho_\alpha}\} \quad \text{for } 2 \le j \le k. \tag{5.9}$$

From (3.17) and (5.9), the composition $A'\partial_2$, as a $(k-1) \times k$ matrix, takes the form

$$A'\partial_2 = \left(0, A^1_{\rho_\Gamma\otimes\alpha}\right) : l^2(\Gamma)^{[k-1]}_{\rho_\alpha} \to l^2(\Gamma)^{[k]}_{\rho_\alpha}. \tag{5.10}$$

Note that $\Psi(x_1 - 1)$ is injective and has dense image. So, A' is also injective and has dense image by (5.9). By (5.10), we have ∂_2 is injective if and only if $A^1_{\rho_\Gamma\otimes\alpha}$ is injective. This proves that $(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ (and thus $(C_*(S^3\backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$) is weakly acyclic if and only if $A^1_{\rho_\Gamma\otimes\alpha}$ is injective.

The method in [12, Sec. 3] does not work directly for the computation of the L^2-Reidemeister torsion since $\Psi(x_1 - 1)$ may not be invertible. Here, we proceed as follows.

For any $\varepsilon > 0$, let $A'(\varepsilon) : l^2(\Gamma)^{[k]}_{\rho_\alpha} \to l^2(\Gamma)^{[k]}_{\rho_\alpha}$ be the $k \times k$ matrix such that

$$\begin{aligned} A'_{1,1}(\varepsilon) &= \Psi(x_1) - (1+\varepsilon)\{\mathrm{Id} : l^2(\Gamma_{\rho_\alpha}) \to l^2(\Gamma_{\rho_\alpha})\}, \\ A'_{i,1}(\varepsilon) &= \Psi(x_i) - \{\mathrm{Id} : l^2(\Gamma)_{\rho_\alpha} \to l^2(\Gamma)_{\rho_\alpha}\}, \quad 2 \le i \le k, \\ A'_{i,j}(\varepsilon) &= \delta_{ij}\{\mathrm{Id} : l^2(\Gamma_{\rho_\alpha}) \to l^2(\Gamma_{\rho_\alpha})\}, \quad 2 \le j \le k. \end{aligned} \tag{5.11}$$

Clearly, $A'(\varepsilon)$ is invertible for $\varepsilon > 0$.

Let $(C_{*,\varepsilon}(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ be the chain complex

$$0 \to l^2(\Gamma)^{[k-1]}_{\rho_\alpha} \xrightarrow{\partial_2} l^2(\Gamma)^{[k]}_{\rho_\alpha,\varepsilon} \xrightarrow{\partial_1} l^2(\Gamma)_{\rho_\alpha} \to 0, \tag{5.12}$$

where $l^2(\Gamma)^{[k]}_{\rho_\alpha,\varepsilon}$ admits the new inner product $\langle \cdot, \cdot \rangle_\varepsilon$ given by

$$\langle x, y \rangle_\varepsilon = \langle A'(\varepsilon)^* A'(\varepsilon) x, y \rangle. \tag{5.13}$$

By (3.25) and (5.11), we have, for any $\varepsilon > 0$,

$$\mathrm{Det}_\tau(A'(\varepsilon)) = 1 + \varepsilon. \tag{5.14}$$

By (5.12)–(5.14) and [2, Proposition 3.11],

$$(1+\varepsilon) T^{(2)}(C_{*,\varepsilon}(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha}) = T^{(2)}(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha}). \tag{5.15}$$

We now examine $T^{(2)}(C_{*,\varepsilon}(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ and its limit as $\varepsilon \to 0^+$. Let $\partial^*_{1,\varepsilon}$ be the adjoint of ∂_1 with respect to the new inner product (5.13) in $(C_{*,\varepsilon}(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$. Thus, we get

$$\begin{aligned} \langle \partial_1 x, y \rangle = \langle x, \partial_1^* y \rangle &= \left\langle (A'(\varepsilon)^* A'(\varepsilon))^{-1} x, \partial_1^* y \right\rangle_\varepsilon \\ &= \left\langle x, (A'(\varepsilon)^* A'(\varepsilon))^{-1} \partial_1^* y \right\rangle_\varepsilon, \end{aligned} \tag{5.16}$$

and $\partial^*_{1,\varepsilon} = (A'(\varepsilon)^* A'(\varepsilon))^{-1} \partial_1^* = A'(\varepsilon)^{-1} (A'(\varepsilon)^*)^{-1} \partial_1^*$. Therefore,

$$\partial_1 \partial^*_{1,\varepsilon} = \partial_1 A'(\varepsilon)^{-1} (A'(\varepsilon)^*)^{-1} \partial_1^*. \tag{5.17}$$

By (5.11), one deduces directly that $A'(\varepsilon)^{-1}$ can be written as

$$\begin{aligned} A'(\varepsilon)^{-1}_{1,1} &= A'_{1,1}(\varepsilon)^{-1}, \\ A'(\varepsilon)^{-1}_{i,1} &= -A'_{i,1}(\varepsilon) A'(\varepsilon)^{-1}_{1,1}, \quad 2 \le i \le k, \\ A'_{i,j}(\varepsilon)^{-1} &= \delta_{ij}\left\{\mathrm{Id} : l^2(\Gamma_{\rho_\alpha}) \to l^2(\Gamma_{\rho_\alpha})\right\}, \quad 2 \le j \le k. \end{aligned} \tag{5.18}$$

From (5.7), (5.11) and (5.18), we have

$$\partial_1 A'(\varepsilon)^{-1} = \underbrace{(\mathrm{Id} + 2\varepsilon A'(\varepsilon)^{-1}_{1,1}, 0, \ldots, 0)}_{k}{}^t - \varepsilon\left(A'(\varepsilon)^{-1}_{1,1}, \ldots, A'(\varepsilon)^{-1}_{k,1}\right)^t. \tag{5.19}$$

By (5.11) and (5.17)–(5.19),

$$\begin{aligned}
&\partial_1\partial_{1,\varepsilon}^* \\
&\quad= \left(\mathrm{Id} + \varepsilon A'(\varepsilon)_{1,1}^{-1}\right)^* \left(\mathrm{Id} + \varepsilon A'(\varepsilon)_{1,1}^{-1}\right) + \varepsilon^2 \sum_{j=2}^{k} \left(A'(\varepsilon)_{j,1}^{-1}\right)^* A'(\varepsilon)_{j,1}^{-1} \\
&\quad= \left(A'(\varepsilon)_{1,1}^{-1}\right)^* \left(\Psi(x_1-1)^*\Psi(x_1-1) + \varepsilon^2 \sum_{j=2}^{k} \Psi(x_j-1)^*\Psi(x_j-1)\right) \left(A'(\varepsilon)_{1,1}^{-1}\right).
\end{aligned} \tag{5.20}$$

By (3.25), (5.11) and (5.18), one has that for any $\varepsilon > 0$,

$$\mathrm{Det}_\tau\left(A'(\varepsilon)_{1,1}^{-1}\right) = \frac{1}{1+\varepsilon}. \tag{5.21}$$

From (2.5), (2.7), Lemma 3.1, Proposition 3.2, (5.20) and (5.21),

$$\lim_{\varepsilon\to 0^+} \mathrm{Det}_\tau\left(\partial_1\partial_{1,\varepsilon}^*\right) = 1. \tag{5.22}$$

Similarly, by (5.13),

$$\langle \partial_2 x, y\rangle_\varepsilon = \langle A'(\varepsilon)^* A'(\varepsilon)\partial_2 x, y\rangle = \langle x, \partial_2^* A'(\varepsilon)^* A'(\varepsilon) y\rangle, \tag{5.23}$$

and the adjoint of ∂_2 with respect to $\langle\cdot,\cdot\rangle_\varepsilon$ is given by

$$\partial_{2,\varepsilon}^* = \partial_2^* A'(\varepsilon)^* A'(\varepsilon). \tag{5.24}$$

By (5.9), (5.10) and (5.24), we have

$$\begin{aligned}
\partial_{2,\varepsilon}^*\partial_2 &= \partial_2^* A'(\varepsilon)^* A'(\varepsilon)\partial_2 \\
&= A^1_{\rho_\Gamma\otimes\alpha}\left(A^1_{\rho_\Gamma\otimes\alpha}\right)^* + \varepsilon^2 \left(\Psi\left(\frac{\partial r_1}{\partial x_1}\right), \ldots, \Psi\left(\frac{\partial r_{k-1}}{\partial x_1}\right)\right)^t \\
&\quad\times \left(\Psi\left(\frac{\partial r_1}{\partial x_1}\right)^*, \ldots, \Psi\left(\frac{\partial r_{k-1}}{\partial x_1}\right)^*\right).
\end{aligned} \tag{5.25}$$

From (2.5) and (5.25),

$$\lim_{\varepsilon\to 0} \mathrm{Det}_\tau\left((\partial_2\partial_{2,\varepsilon}^*)|_{\overline{\mathrm{Im}(\partial_2)}}\right) = \lim_{\varepsilon\to 0} \mathrm{Det}_\tau(\partial_{2,\varepsilon}^*\partial_2) = \mathrm{Det}_\tau(A^1_{\rho_\Gamma\otimes\alpha})^2. \tag{5.26}$$

By (4.2), (5.15), (5.22) and (5.26), we obtain

$$T^{(2)}(C_*(W, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha}) = \frac{1}{\mathrm{Det}_\tau(A^1_{\rho_\Gamma\otimes\alpha})}, \tag{5.27}$$

and the result follows. ◻

Remark 5.2. For $t = 1$, Proposition 5.1 is due to Lück [17] (cf. [16, Theorem 4.9]), with a different proof. Moreover, he showed that the chain complex $(C_*(S^3\backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is both weakly acyclic and of determinant class for $t = 1$. We will extend these results to the $t \in U(1)$ case in the next section.

6. A Rigidity Property for the $U(1)$ Twisted L^2-Torsion on a Knot Complement

The main result of this section is the following.

Theorem 6.1. *For any knot K in S^3, the complex $(C_*(S^3\backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ is both weakly acyclic and of determinant class. Moreover, the L^2-Reidemeister torsion $T^{(2)}(C_*(S^3\backslash K, l^2(\Gamma)_{\rho_\alpha}), \partial_{\rho_\alpha})$ does not depend on the unitary representation α in* (3.3).

In view of Proposition 5.1, this result can be reformulated as follows.

Theorem 6.1′. *For $t \in U(1)$, every $A^j_{\rho_\Gamma\otimes\alpha}(t)$, $1 \le j \le k$, defined in* (3.9) *is both injective and of determinant class. Moreover,*

$$\mathrm{Det}_\tau(A^j_{\rho_\Gamma\otimes\alpha}(t)) = \mathrm{Det}_\tau(A^j_{\rho_\Gamma\otimes\alpha}(1)). \tag{6.1}$$

Proof. Without loss of generality, we assume $j = 1$. Simply denote $A^1_{\rho_\Gamma\otimes\alpha}(t)$ by A_t. Let A_t^* be the adjoint operator of A_t and $\overline{t}$ be the conjugate of t. Observe first that for $t \in U(1)$, one has

$$t^{-1} = \overline{t}. \tag{6.2}$$

Let L be a positive constant such that for any $t \in U(1)$,

$$L \ge \|A_t\|, \tag{6.3}$$

where $\|A_t\|$ is the operator norm of A_t. The existence of L is clear.

Following [16, Definition 3.171], for any positive integer p, set

$$c(A_t, L)_p = \mathrm{Tr}_\tau[(\mathrm{Id} - L^{-2}A_t^*A_t)^p]. \tag{6.4}$$

Lemma 6.2. *For any positive integer p, we have*

$$c(A_t, L)_p = c(A_1, L)_p. \tag{6.5}$$

Proof. By the definition of A_t, it is clear that each matrix element of A_t can be expressed as a linear sum with real coefficients of elements of the form

$$t^{j_1+\cdots+j_s}\rho_\Gamma(\gamma_{i_1}^{j_1}\cdots\gamma_{i_s}^{j_s}), \quad j_m \in \mathbf{Z} \quad \text{for } 1 \le m \le s, \tag{6.6}$$

where each $\gamma_{i_m} = \phi(x_{i_m})$ $(1 \le m \le s)$ with x_{i_m} being one of the generators $x_1, \ldots, x_k$ (recall that ϕ is the natural map from the free group F_k to Γ).

Since the fundamental representation ρ_Γ of Γ is unitary, from (6.2), one deduces that the matrix elements in the matrix form of A_t^* are of the same form, and so are the matrix elements in the matrix form of $(\mathrm{Id} - L^{-2}A_t^*A_t)^p$.

Let $c_{p,i}(t)$ $(1 \le i \le k-1)$ be the diagonal elements of the the matrix form of $(\mathrm{Id} - L^{-2}A_t^*A_t)^p$. Then, each $c_{p,i}(t)$ $(1 \le i \le k-1)$ can be expressed as a linear sum with real coefficients of elements of the form (6.6).

For any term of the form (6.6), we claim that the von Neumann trace

$$\mathrm{Tr}_\tau[t^{j_1+\cdots+j_s}\rho_\Gamma(\gamma_{i_1}^{j_1}\cdots\gamma_{i_s}^{j_s})]$$

vanishes if $j_1+\cdots+j_s \neq 0$. For if $j_1+\cdots+j_s \neq 0$, then $\gamma_{i_1}^{j_1}\cdots\gamma_{i_s}^{j_s}$ maps to $h^{j_1+\cdots+j_s}$, where h is the generator of $H_1(S^3\backslash K, \mathbf{Z})$, under the natural map $\Gamma = \pi_1(S^3\backslash K) \to H_1(S^3\backslash K, \mathbf{Z})$. This implies that $\gamma_{i_1}^{j_1}\cdots\gamma_{i_s}^{j_s} \in \Gamma$ is not the unit element. By (2.2), the above von Neumann trace vanishes.

Note that for each $1 \leq i \leq k-1$, the elements of the form (6.6) with $j_1+\cdots+j_s = 0$ in $c_{p,i}(t)$ are exactly the same elements with possible non-vanishing von Neumann trace appearing in $c_{p,i}(1)$. By (2.2), we get

$$\mathrm{Tr}_\tau[c_{p,i}(t)] = \mathrm{Tr}_\tau[c_{p,i}(1)]. \tag{6.7}$$

Hence, our result follows:

$$c(A_t, L)_p = \sum_{i=1}^{k-1} \mathrm{Tr}_\tau[c_{p,i}(t)] = \sum_{i=1}^{k-1} \mathrm{Tr}_\tau[c_{p,i}(1)] = c(A_1, L)_p. \tag{6.8}$$

□

By [16, Definition 2.1] and [16, Theorem 3.172(2)], A_t is injective if and only if

$$\lim_{p\to+\infty} c(A_t, L)_p = 0. \tag{6.9}$$

For the injective A_t, we have that A_t is of determinant class if and only if the sum of positive real numbers $\sum_{p=1}^{+\infty} \frac{c(A_t,L)_p}{p}$ converges by [16, Theorem 3.172(2) and (4)]. Moreover,

$$\log(\mathrm{Det}_\tau(A_t)) = (k-1)\log(L) - \sum_{p=1}^{+\infty} \frac{c(A_t, L)_p}{2p}. \tag{6.10}$$

In [17] (cf. [16, Theorem 4.9]), Lück proved that A_1 is injective and of determinant class. By combining this result with Lemma 6.2, (6.9) and (6.10), we complete the proof of Theorem 6.1′. □

Remark 6.3. Theorems 6.1 and 6.1′ show that in contrast with the usual Alexander polynomial or the Reidemeister torsion of the knot complement, the L^2-counterpart has a strong rigidity property. Initially this may sound strange. If one observes that when K is a hyperbolic knot, then the L^2-Reidemeister torsion of the knot complement determines the hyperbolic volume of the knot complement [19, Theorem 0.6] (cf. [16, Theorem 4.3]). The above result would imply certain rigidity of the hyperbolic volume.

Remark 6.4. All the discussions above also holds if one replaces ρ_Γ by $\rho_\Gamma \otimes \beta$ with a unitary representation β of Γ. In order to get possible nontrivial deformations of the L^2-Alexander invariant, we have to consider non-unitary representations of Γ. This will be briefly dealt with in the next section.

7. The L^2-Alexander Invariant Associated to $\mathbf{C}^*$ Representations

In this section, we first replace the representation α in (3.3) by

$$\alpha' : \Gamma \to \mathbf{C}^* = \mathbf{C}\backslash\{0\}, \quad \alpha'(x_j) = t \in \mathbf{C}^*\backslash U(1), \quad j = 1, \dots, k. \tag{7.1}$$

By proceeding as in Sec. 3, one sees that $\mathrm{Det}_\tau(A^1_{\rho_\Gamma \otimes \alpha'})$ is well defined up to the multiplicative group $\{|t|^p\}_{p\in\mathbf{Z}}$. If $A^1_{\rho_\Gamma\otimes\alpha'}$ is injective and of determinant class, then we define an L^2-Alexander type invariant $\Delta^{(2)}_K(t)' \in \mathbf{R}/\mathbf{Z}$ by

$$\Delta^{(2)}_K(t)' \equiv \frac{\log(\mathrm{Det}_\tau(A^1_{\rho_\Gamma\otimes\alpha'}))}{\log(|t|)} \quad (\mathrm{mod}\ \mathbf{Z}). \tag{7.2}$$

Let $q : \mathbf{C}^* \to \{0, 1\}$ be the function defined by

$$q(t) = -\min\{\mathrm{sgn}(|t| - 1), \mathrm{sgn}(1 - |t|), 0\}. \tag{7.3}$$

Then, we define

$$\Delta^{(2)}_K(t)' \equiv \frac{\log(\mathrm{Det}_\tau(A^1_{\rho_\Gamma\otimes\alpha'}))}{(\log(|t|))^{q(t)}} \quad (\mathrm{mod}\ q(t)\mathbf{Z}), \tag{7.4}$$

where $q(t) = 0$ for $t \in U(1)$ (we use the convention that $x^0 \equiv 1$). Thus, for $t \in U(1)$, we have

$$\Delta^{(2)}_K(t)' = \log(\Delta^{(2)}_K(t)), \tag{7.5}$$

which by Theorem 6.1′ does not depend on $t \in U(1)$. This provides a uniform way to define our L^2-Alexander invariant in Sec. 3 and (7.2).

Theorem 7.1. *The L^2-Alexander invariant $\Delta^{(2)}_K(t)'$ depends only on $|t|$.*

Proof. We have proved the case for $t \in U(1)$. For general $t \in \mathbf{C}^*$, we write $t = |t|u$. Then, $\bar{t} = |t|u^{-1}$. The result follows from the same argument in Lemma 6.2. □

Example 7.2. Since $\pi_1(S^3\backslash\text{unknot}) = \mathbf{Z}$, it is easy to verify directly that

$$\Delta^{(2)}_{\text{unknot}}(t)' = 0 \quad (\mathrm{mod}\ q(t)\mathbf{Z}). \tag{7.6}$$

Example 7.3. We consider the figure eight knot 4_1. Then, $\Gamma = \pi_1(S^3\backslash 4_1)$ has a presentation

$$P(\Gamma) = \langle x, y \,|\, zxz^{-1}y^{-1}\rangle \quad \text{with } z = x^{-1}yxy^{-1}x^{-1}. \tag{7.7}$$

By applying the Fox derivative to $r : zxz^{-1}y^{-1} = 1$, one gets (cf. [7, Example 4.1])

$$\frac{\partial r}{\partial x} = -x^{-1} + x^{-1}y + yx^{-1} - yx^{-1}y + yx^{-1}yxy^{-1}x^{-1}. \tag{7.8}$$

If we briefly write ρ_Γ for $\rho_\Gamma \circ \phi$, then we have

$$\begin{aligned} A_t &= \Psi\left(\frac{\partial r}{\partial x}\right) \\ &= -t^{-1}\rho_\Gamma(x^{-1}) + \rho_\Gamma(x^{-1}y) + \rho_\Gamma(yx^{-1}) - t\rho_\Gamma(yx^{-1}y) + \rho_\Gamma(yx^{-1}yxy^{-1}x^{-1}) \\ &= (\mathrm{Id} - t\rho_\Gamma(x^{-1}yx) - t\rho_\Gamma(y) - t\rho_\Gamma(yx^{-1}yxy^{-1}) \\ &\quad + t^2\rho_\Gamma(yx^{-1}yx))(-t^{-1}\rho_\Gamma(x^{-1})). \end{aligned} \tag{7.9}$$

It is clear that $t \in \mathbf{C}$ with $|t| < \frac{1}{4}$,

$$\|t\rho_\Gamma(x^{-1}yx) + t\rho_\Gamma(y) + t\rho_\Gamma(yx^{-1}yxy^{-1}) - t^2\rho_\Gamma(yx^{-1}yx)\| < 1 \tag{7.10}$$

so that $\mathrm{Id} - t\rho_\Gamma(x^{-1}yx) - t\rho_\Gamma(y) - t\rho_\Gamma(yx^{-1}yxy^{-1}) + t^2\rho_\Gamma(yx^{-1}yx)$ is invertible. By proceeding as in the proof of (3.25) and (6.5), we obtain

$$\mathrm{Det}_\tau(\mathrm{Id} - t\rho_\Gamma(x^{-1}yx) - t\rho_\Gamma(y) - t\rho_\Gamma(yx^{-1}yxy^{-1}) + t^2\rho_\Gamma(yx^{-1}yx)) = 1. \tag{7.11}$$

By (7.4), (7.9), (7.11) and (2.6), (2.7),

$$\Delta^{(2)}_{4_1}(t)' = 0 \pmod{\mathbf{Z}}. \tag{7.12}$$

Similarly, (7.12) holds for $t \in \mathbf{C}$ with $|t| > 4$.

On the other hand, by a direct computation of the hyperbolic volume of $S^3\backslash 4_1$ (cf. [10, (3.17)]) and [19, Theorem 0.6] (cf. [16, Theorem 4.3]), one finds

$$\Delta^{(2)}_{4_1}(1)' = \log(\Delta^{(2)}_{4_1}(1)) = \frac{\mathrm{vol}(S^3\backslash 4_1)}{6\pi} \sim \frac{1}{6\pi} \cdot 2.029 \neq 0 \pmod{\mathbf{Z}}. \tag{7.13}$$

Example 7.3 shows that $\Delta^{(2)}_K(t)'$ is not a constant by considering non-unitary representations, and gives a nontrivial deformation of the L^2-torsion of the knot complement. If K is a hyperbolic knot, then $\{\Delta^{(2)}_K(t)' : t \in \mathbf{R}^*\}$ forms a nontrivial deformation of the hyperbolic volume of $S^3\backslash K$. It will be interesting to study further properties of the distribution of this invariant over $\mathbf{R}^*$.

Remark 7.4. In view of [7, Theorem 3.1], one may regard (7.12) as a monic property of the L^2-Alexander invariant $\Delta^{(2)}_K(t)'$. By Remark 3.6, the L^2-Alexander invariant $\Delta^{(2)}_K(t)'$ can be defined for any presentation of Γ which is strongly Tietze-equivalent to a Wirtinger one. This may enlarge the applicability of this invariant.

Let $P(\Gamma)$ be a presentation of Γ, which is strongly Tietze-equivalent to a Wirtinger one, given by

$$P(\Gamma) = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1}\rangle. \tag{7.14}$$

Let $\rho_\alpha : H_1(S^3\backslash K) \to \mathbf{C}^*$ be a representation of $H_1(S^3\backslash K)$. If the natural image of $\phi(x_j)$ $(1 \le j \le k)$ in $H_1(S^3\backslash K)$ is h^{j_s} with $j_s \in \mathbf{Z}$, then we may replace (3.3) by

$$\alpha(x_j) = t^{j_s}. \tag{7.15}$$

Now, one can proceed as in Sec. 3 without any change. In particular, one can construct $A^j_{\rho_\Gamma\otimes\alpha}$ in the same way. Moreover, if there is one $j \in \mathbf{N}$ with $1 \le j \le k$ and $j_s \neq 0$ verifying that $A^j_{\rho_\Gamma\otimes\alpha}$ is injective and of determinant class, then for every

$j \in \mathbf{N}$ with $1 \leq j \leq k$ and $j_s \neq 0$, $A^j_{\rho_\Gamma \otimes \alpha}$ is injective and of determinant class. Define $\Delta^{(2)}_K(t)'$, in this case, to be

$$\Delta^{(2)}_K(t)' \equiv \frac{\log(\mathrm{Det}_\tau(A^j_{\rho_\Gamma \otimes \alpha}))}{(\log(|t|))^{q(t)}} \mod q(t)\mathbf{Z}. \tag{7.16}$$

Example 7.5. Let $m > 1$ and $n > 1$ be a pair of coprime integers. Let $T_{m,n}$ be the corresponding torus knot. Then, $T_{m,n}$ admits a presentation

$$P(\pi_1(T_{m,n})) = \{x_1, x_2 \,|\, x_1^n = x_2^m\}. \tag{7.17}$$

Moreover, the natural image of x_1 (resp. x_2) in $H_1(S^3 \backslash T_{m,n})$ is h^m (resp. h^n). We assume that this presentation is strongly Tietze-equivalent to a Wirtinger presentation of $\pi_1(T_{m,n})$ (a typical example is the trefoil knot).

By a direct computation of the Fox derivative, one has

$$\frac{\partial r}{\partial x_1} = x_1^{n-1} + x_1^{n-2} + \cdots + x_1 + 1. \tag{7.18}$$

Note that $\Psi(x_1^n - 1) = \Psi\big(x_1^{n-1} + x_1^{n-2} + \cdots + x_1 + 1\big)\Psi(x_1 - 1)$. We have $\Psi(x_1^{n-1} + x_1^{n-2} + \cdots + x_1 + 1)$ is injective by Remark 3.3. Moreover,

$$\mathrm{Det}_\tau(\Psi(x_1^{n-1} + x_1^{n-2} + \cdots + x_1 + 1)) = \max(1, |t|^{mn-m}). \tag{7.19}$$

By (7.16) and (7.19), for any $t \in \mathbf{C}^*$, we have

$$\Delta^{(2)}_{T_{m,n}}(t)' = 0 \mod q(t)\mathbf{Z}. \tag{7.20}$$

Remark 7.6. It should be interesting to give an L^2-torsion interpretation of $\Delta^{(2)}_K(t)'$ for the case of $|t| \neq 1$.

Remark 7.7. Another possible non-unitary extension is to stay with the representation α in (3.3) but replace the unitary representation β of Γ appeared in Remark 6.4 by a unimodular one. Then, by proceeding as in Sec. 3, one obtains an intrinsic invariant

$$\Delta^{(2)}_{K,\beta}(t) = \frac{\mathrm{Det}_\tau(A^j_{\rho_\Gamma \otimes \beta \otimes \alpha})}{\mathrm{Det}_\tau(\Psi_\beta(x_j - 1))}, \tag{7.21}$$

where Ψ_β is simply the Ψ in Sec. 3 with ρ_Γ replaced by $\rho_\Gamma \otimes \beta$, when both $A^j_{\rho_\Gamma \otimes \beta \otimes \alpha}$ and $\Psi_\beta(x_j - 1)$ are injective and $\mathrm{Det}_\tau(\Psi_\beta(x_j - 1)) \neq 0$. We believe that these invariants would not be constant even for $t \in U(1)$, though the actual computation is not easy.

8. Possible Relation with the Volume Conjecture

Let K be a knot in S^3 and let $J_N(K, q)$, $N \in \mathbf{N}$, denote the normalized colored Jones polynomial of K (cf. [21, 24]). By Proposition 5.1 and Theorem 6.1, the volume conjecture of Kashaev [12] and, Murakami and Murakami [24] can be restated as follows (cf. [16, Conjecture 4.8]),

$$\lim_{N\to+\infty}\left|J_N\left(K,\exp\left(\frac{2\pi\sqrt{-1}}{N}\right)\right)\right|^{\frac{1}{3N}}=\Delta_K^{(2)}(1). \tag{8.1}$$

By comparing (8.1) with the Melvin–Morton conjecture stated in [21] (the Melvin–Morton conjecture was proved formally in [25] and rigorously in [1], see also [14] for an alternate proof), it seems plausible to view the volume conjecture as a kind of L^2-analogue of the Melvin–Morton conjecture. This fits with the picture outlined by Gukov in [8]. In particular, the rigidity property in Theorem 6.1 fits with the form of the generalized Melvin–Morton conjecture stated in [8], where the hyperbolic torsion in the right-hand side of [8, (6.30)] (which should play a role of the L^2-torsion (or the L^2-Alexander invariant) here) does not contain a (unitary) deformed parameter.

A possible way to approach (8.1) may be first to construct certain L^2-analogue of the (colored) Jones polynomial. Then, one can try to give a Chern–Simons theory interpretation of the L^2-Alexander invariant,[b] as well as the potential L^2-Jones invariant,[c] and then try to produce an L^2-analogue of [25].

On the other hand, as the L^2-invariant $\Delta_K^{(2)}(t)'$ deforms the right-hand side of (8.1), it seems natural to ask whether there are suitable deformed versions of the left-hand side of (8.1).

Consider now the fundamental representation

$$\rho'_\Gamma : H_1(S^3\backslash K,\mathbf{Z})\to l^2(H_1(S^3\backslash K,\mathbf{Z}))$$

which induces naturally a representation of Γ (denoted by the same notation ρ'_Γ for this extension). Then, we can proceed in exactly the same way as in Sec. 3 to construct an L^2-Alexander invariant $\widetilde{\Delta}_K^{(2)}(t)$ for $t\in U(1)$. It may be thought of as the L^2-Alexander invariant (or the L^2-torsion) associated to the infinite cyclic covering space of the knot complement, instead of the universal covering.

As in Theorem 6.1$'$, one can prove that $\widetilde{\Delta}_K^{(2)}(t)$ does not depend on $t\in U(1)$. Moreover, a direct calculation shows that

$$\widetilde{\Delta}_K^{(2)}(1)=\mathbf{M}(\Delta_K(t)), \tag{8.2}$$

where $\mathbf{M}(\Delta_K(t))$ is the Mahler measure of the usual Alexander polynomial $\Delta_K(t)$ of K. This makes it plausible to ask the following question.

Question 8.1. Whether there is a knot polynomial whose Mahler measure equals to the L^2-Alexander invariant $\Delta_K^{(2)}(1)$ (or equivalently, the L^2-torsion of the knot complement)?

[b] Recall that such an interpretation for the usual Alexander polynomial has been given for example in [26].

[c] Would there be an L^2-analogue of the Chern–Simons–Witten theory [28] for Jones polynomials?

If this question would have a positive answer, then the volume conjecture (8.1) might be stated as a claim between knot polynomials.

Remark 8.2. The identity (8.2) gives a geometric interpretation of the Mahler measure of Alexander polynomials. We hope that it may shed a light on [6, Sec. 4.3, Problem 18].

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